the best interests of the two mathematical organizations in their need of financial support.

Finally Professor Young rendered important service to the Association as a member of the Carus Monograph Committee and in particular as the author of monograph number four on *Projective Geometry* which has received favorable comment from many sources. It was in connection with the preparation and publication of this monograph that the writer came in closest contact with him and learned more completely to admire his many fine qualities.

Professor Young was a staunch supporter of the Mathematical Association of America and his great desire was to see its influence still further widened. To this thesis he directed his retiring presidential address in which he set forth a plan calculated to bring the Association into closer touch with undergraduates in our colleges and universities. That plan, Professor Young realized, is not feasible at present on account of the necessary financial support which it would involve. However, it is believed that some improvements along this line can be effected, at least partially, through modifications of certain departments of the Monthly. The Board of Editors have these matters under consideration and definite announcement may be expected in the near future. Meanwhile a rereading of Professor Young's presidential address may serve to inspire us all with his spirit and to spur us on to more effective service in the cause which was so dear to him and for which, in a sense, he gave his life.

His Life and Scientific Activities by L. L. Silverman

John Wesley Young died at Hanover on February 17, 1932 at the early age of fifty-two. His death brought a feeling of profound sorrow throughout the country; for he was not only a devoted friend to his colleagues in Hanover, but a personal friend to many mathematicians throughout the land.

He was born November 17, 1879, at Columbus, Ohio. His father, William Henry Young, was a college professor and later an American consul in Baden-Baden, Germany. His mother, Marie Louise Widenhorn Young, was born in Paris, of French and German ancestry. In 1907 Professor Young was married to Mary Louise Aston of Columbus, Ohio. Their daughter, Mary Elizabeth, was graduated from Smith College in 1930 and was married in May of this year.

Professor Young received most of his early education in Germany. During the years of his father's consular service in Baden-Baden, 1889–1895, he attended the German Gymnasium and prepared himself to enter the university. He attended the institution in his native city and was graduated from Ohio State University in 1899. He then went for his graduate work to Cornell, where he received the doctor's degree in 1904. He taught at Northwestern, Princeton, University of Illinois, University of Kansas, University of Chicago and Dartmouth College. He came to Dartmouth in 1911 as head of the department of mathematics.

He was active in research; but he was also interested in the problems of teaching both elementary and advanced mathematics. He was appointed to many positions to which he devoted his energies and ability; and he was elected to positions of honor as a result of the recognition of the importance of his achievements. He was editor of the Bulletin, 1907–1925; on the editorial board of the Mathematics Teacher, 1921–1928; chief examiner in geometry, College Entrance Examination Board, 1913–1915. When in 1916 the Mathematical Association of America organized the National Committee on Mathematical Requirements for the purpose of investigating possible reforms in the teaching of secondary school mathematics, Professor Young was appointed chairman of this committee. To this work he devoted eight years, accomplishing results which will long affect American mathematics. He was also editor of a series of mathematical texts published by Houghton, Mifflin and Company. Professor Young was president of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England, 1926–1927; vice-president of the American Mathematical Society, 1928–1930; and president of the Mathematical Association of America, 1929–1930. He was a member of Phi Beta Kappa and Sigma Xi, and of many learned societies both here and abroad.

He began his mathematical research[1] in the subject of groups, in which he always retained a keen interest. But his interest continually widened. He soon began to work in projective geometry and in logical foundations; and in 1910 he wrote, in collaboration with Professor Veblen, the well-known book on projective geometry, building the geometry upon a set of independent axioms, which were quite different from any that had hitherto been used. It was Professor Young's opinion that the axiomatic method was the best way to teach projective geometry in the university. Nevertheless he later wrote for the non-specialist an excellent book on projective geometry, the Carus Mathematical Monograph, No. 4, where the axiomatic method was entirely abandoned.

One paper written by Professor Young about twenty-five years ago has been widely quoted in the literature and is still highly spoken of by specialists in the field. This paper is entitled *General theory of approximations by functions involving a given number of arbitrary parameters*, and appeared in the Transactions of the American Mathematical Society in 1907. Tchebychev was the first to consider the problem of determining the polynomial $p(x)$ of given degree n which gives the closest approximation to the given function $\phi(x)$ on the interval (a, b). Tchebychev defines the approximation to be the "closest" when the maximum of $f(x) = |p(x) - \phi(x)|$ in the interval (a, b) is as small as possible. Professor Young considers the following more general problem. Let the class C contain $g(x; a_0, a_1, a_2, \cdots, a_n)$, a function of x and the parameters $a_0, a_1, \cdots, a_n$; and let f be identified with $U(g)$, the result of operating on g with some functional

[1] A complete bibliography of the books and articles by Professor Young is being prepared for publication.

operation U. If there exists a set of parameters a_i which renders the maximum of $U(g)$, as x varies over a given finite interval, as small as possible, the resulting function g is called a function of approximation in the class C with reference to U. It is then shown that under very general conditions, such a function of approximation exists, and a uniqueness theorem is obtained.

Professor Young felt that there should be in our colleges a course on the history of science, which would give the non-specialist the fundamental ideas of science as they developed through the ages. He urged the introduction of such a course at Dartmouth and himself undertook to teach the course for the first few years. It required a wide knowledge of science and much labor in preparation. The course which resulted was most valuable and stimulating to the students who were in it.

Professor Young's mathematical outlook may be described by four characteristics:

He was interested in the *unity* of mathematics. His was not a narrow interest in some special field. He listened with sympathy and understanding to ideas in every field; and often through his natural keenness he was able to make suggestions to specialists in fields far removed from his own. But while having an interest in and an understanding of a wide field of mathematical subjects, he always felt that there was an inner unity in mathematics. And this feeling applied also to the elementary field, where in his book, *Elementary Mathematical Analysis* (written in collaboration with F. M. Morgan), he tried to relate all the ideas of algebra, trigonometry and analytic geometry to the central notion of function.

He was interested in the process of *generalization*. Moore's General Analysis appealed to him greatly. The article on the general theory of approximation, referred to above, is an example of his desire for generalization. Another illustration is his article, *A new formulation for general algebra*, which appeared in the Annals in 1927. Nevertheless he did not approve of mere generalization as such; the generalization must be significant. In his retiring presidential address to this Association he refers to the generalization of the idea of continuity. He goes on to say: "The means employed was the concept of the 'development' of the range of the general variable. On the basis of this concept he (Professor Pitcher) arrived very naturally at the definition of 'continuity with respect to a given development.' He then proved the surprising theorem that *every function is continuous with respect to some development of the range*. The result may be of interest—I am inclined to think it is—but as a generalization of continuity it is obviously futile."

What appealed to Professor Young most, perhaps, was the *beauty* of mathematics. He often spoke of mathematics as a fine art; and like Professor Bôcher, he saw in the combination of the creative and imaginative activities of the mathematician something which resembled the activities of the artist. On a number of occasions he made popular addresses on this subject, taking his illus-

trations largely from the introduction of ideal elements in projective geometry.

The *popularization* of mathematics was an idea close to Professor Young's heart. Here is a subject full of beautiful ideas, relations and generalizations. Would it not be splendid to eliminate the technical side so far that these beautiful things might be appreciated also by those students who do not specialize in mathematics? In answer to this question, his book of *Lectures on the fundamental concepts of algebra and geometry* is a masterpiece. He was particularly interested in the Synoptic Course at Dartmouth, a survey course endeavoring to give to the non-specialist, with the minimum of technique, some understanding and appreciation of a number of topics in advanced mathematics.

Professor Young lived the life of a philosopher. He was fearless in his convictions, sympathetic to his friends, fairminded to his opponents. These features were such an outstanding part of his character that they exerted a marked influence on all who came in contact with him. His life was regrettably short. A consolation to his friends is that it was a very happy one.

EDITORIAL POLICY

The American Mathematical Monthly is "devoted to the interests of collegiate mathematics"—if one accepts the statement on the cover. But if one looks further than the cover page, he may find reason to doubt the whole-heartedness of this devotion. There has been a rapidly increasing demand for space in those American journals whose function is the publication of the results of mathematical research; and the devotion of the Monthly to collegiate mathematics has not been so complete as to leave it entirely indifferent to the attractions in this somewhat different direction. It has been, and is being, tempted.

In his retiring presidential address which appeared in the January issue of the Monthly, the late Professor Young called attention to the "marked shift in the appeal" of the Monthly in recent years; and he suggested the establishment of a "new magazine devoted to the mathematical interests of undergraduates." But the necessary funds and facilities are not available for such an undertaking, and for the present the Monthly must see to it that this field is not entirely neglected.

While no sharp change of policy is contemplated, it is the desire of the editors that every number of the Monthly shall contain a certain amount of material that will be of interest to the more capable undergraduate. Good expository papers will be especially acceptable—papers which begin with ideas easily understood by the better college senior and then lead him on into unfamiliar fields. Some research papers will be accepted as they have been in the past, but subject to certain restrictions as to length and style of presentation. It is clear that

a style suitable for a paper in the Transactions, for instance, assuming on the part of the reader a general knowledge of, and easy access to, the literature of the subject, familiarity with technical language and symbols, and a high degree of mathematical maturity and experience, would not be a style suitable for the presentation of the same material to the readers of the Monthly.

But it is in the departments of *Problems and Solutions*, *Questions andDiscussions*, and *Mathematics Clubs* that we have our best opportunity to offer mathematical stimulant to our hypothetically thirsty undergraduate. Beginning this fall, probably with the October number, we expect to subdivide the department of *Problems and Solutions* into *Elementary Problems* and *More Difficult Problems*. Professor W. F. Cheney of the Connecticut Agricultural College will take charge of the division of elementary problems. He is hoping for the cooperation of all our readers in securing a steady supply of suitable material. The problems should be relatively easy without being too trivial, and should have, as far as possible the sort of novelty that will arouse interest. Much material will doubtless be sent in which, for one reason or another, can not be used; but it is hoped that this will not offend or discourage contributors.

The department of *Questions and Discussions* is to be expanded into *Questions, Discussions, and Comments* in order to allow somewhat more latitude in the type of papers included; and the department will use some papers of quite elementary character when they appear to be appropriate and sufficiently interesting.

In the department of *Mathematics Clubs* we welcome not only the reports of club activities, but material suitable for club programs.

There is need for the cooperation of our readers in another direction. Very few college undergraduates subscribe to the Monthly. It will reach them only as college teachers call their attention to it, and make the issues regularly available in a library or otherwise. Institutional membership in the Association provides two copies of the Monthly, one for the library and one for the individual use of mathematics club members. If teachers will, from time to time call attention to particular problems or discussions that might possibly interest their better students, we can hope to function to some extent as a stimulus to undergraduate mathematical activity.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-nine persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

G. A. BAKER, Ph.D. (Illinois) Head of Dept., Shurtleff Coll., Alton, Ill.

C. J. BLACKALL, M. S. (Notre Dame) Instr., Math. and Physics, St. Thomas Coll., St. Paul, Minn.

J. L. BOTSFORD, A.B. (Washington) Teaching fellow, California Inst. of Tech., Pasadena, Calif.

A. L. BUCKMAN, A.M. (California) Teaching fellow, Univ. of California, Berkeley, Calif.

IRIS CALLAWAY, A.M. (Peabody) Asso. Prof., State Teachers Coll., Athens, Ga.

A. S. CORBIN, B.S. (Worcester Poly. Inst.) Instr., Coll. of William and Mary, Williamsburg, Va.

D. CROMWELL, A.B. (Baylor) Asst. Prof., John Tarleton Jr. Coll., Stephenville, Tex.

U. P. DAVIS, A.M. (Florida) Instr., Univ. of Florida, Gainesville, Fla.

OLA ESKELSON, M.S. (Illinois) Teacher, High School, Borger, Tex.

J. D. FITZPATRICK, A.B. (Marquette) Instr., Creighton Univ., Omaha, Neb.

EUNICE HUTTO, A.M. (Alabama) Head of Dept., Bob Jones Jr. Coll., College Point, Fla.

LOUISE JOHNSON, A.B. (Colorado) Grad. Asst., Univ. of Colorado, Boulder, Colo.

HARRIS JONES, B.S. (Mass. Inst. of Tech.) Lt. Col., U. S. Army, Prof., U. S. Milit. Acad., West Point, N. Y.

E. R. KELLER, A.M. (Tennessee) Asst., Univ. of Kentucky, Lexington, Ky.

KATE K. KNIGHT, A.B. (East Central S.T.C.) Prof., East Central State Teachers Coll., Ada, Okla.

RUTH G. MASON, Ph.D. (Chicago) Private research, Berkeley, Calif.

POLLY P. NELSON, A.M. (Chicago) Instr., Hunter Coll., New York, N. Y.

W. V. PARKER, Ph.D. (Brown) Prof., Mississippi Woman's Coll., Hattiesburg, Miss.

H. H. PIXLEY, Ph.D. (Chicago) Instr., Coll. of the City of Detroit, Detroit, Mich.

E. S. QUADE, B.S. (Florida) Instr., Univ. of Florida, Gainesville, Fla.

J. H. ROSE, Ph.B. (Wisconsin) Asst., Univ. of Wisconsin, Madison, Wis.

R. G. SANGER, Ph.D. (Chicago) Instr., Univ. of Chicago, Chicago, Ill.

H. L. SCHUG, B.S. (Lafayette) Research Engineer, The Hoover Co., N. Canton, Ohio

V. B. TEACH, M.S. (Ohio State) Asso. Prof., Armour Inst. of Tech., Chicago, Ill.

V. THÉBAULT, Inspecteur d'assurances, Le Mans, France

A. M. WALLACE, M.S. (Oklahoma) Prof., East Central State Teachers Coll., Ada, Okla.

IRENE L. WENTE, M.S. (Chicago) Instr., State Coll., Brookings, S. D.

ADA H. WEST, A.M. (Kansas) Asst. Prof., State Teachers Coll., Peru, Neb.

L. S. WINTON, B.S. (Grove City) Grad. student, Oberlin Coll., Oberlin, Ohio

To Institutional Membership

GEORGIA STATE COLLEGE FOR MEN, Tifton, Ga.

W. D. CAIRNS, *Secretary*

THE NINTH ANNUAL MEETING OF THE MICHIGAN SECTION

The ninth annual meeting of the Michigan Section of the Mathematical Association of America was held in conjunction with the Michigan Academy of Science at the University of Michigan on Saturday, March 19, 1932. Professor J. P. Everett, chairman of the Section, presided.

There were over eighty persons in attendance, including the following forty-four members of the Association: H. M. Ackley, N. H. Anning, W. L. Ayres, J. W. Baldwin, W. D. Baten, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, W. D. Cairns, R. V. Churchill, R. W. Clack, C. C. Craig, S. E. Crowe, Wayne Dancer, Albertus Darnell, L. C. Emmons, C. M. Erikson, J. P. Everett, Peter Field, George Fisanick, K. W. Folley, W. B. Ford, B. C. Getchell, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, C. E. Love, A. L. Nelson, H. L. Olson, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, R. H. Schoonover, E. R. Sleight, C. E. Stout, A. G. Swanson, J. F. Thomson, T. O. Walton, R. L. Wilder, J. B. Winslow.

At the business meeting the following officers were elected for the ensuing year: Chairman, R. W. Clack, Alma College; Secretary-Treasurer, W. L. Ayres, University of Michigan; Member of the Executive Committee, S. E. Crowe, Michigan State College. A motion was made and carried recommending to the new Executive Committee that they consider the advisability of holding two meetings of the Section each year, one, the annual meeting at the University in conjunction with the Michigan Academy of Science, and the other in October at some other institution. A luncheon, attended by fifty-five persons, was held at noon at the Michigan League.

The following papers were presented at the morning meeting:

1. "Evaluation of a certain summation" by Professor W. D. Baten, University of Michigan.

2. "Heat conduction and convection from tall, hot vertical cylinders and high walls at uniform temperature" by Professor W. S. Kimball, Michigan State College, by invitation.

3. "Systems of linear difference equations with constant coefficients" by Professor C. M. Erikson, Michigan State Normal College.

4. "On the Ostogradsky-Hermite method of integration of certain types of functions" by D. K. Kazarinoff, University of Michigan, by invitation.

5. "Edge conditions for double integrals in the calculus of variations" by J. E. Powell, Michigan State College.

6. "Character through mathematics" by Professor R. W. Clack, Alma College.

7. "A mathematical analysis of the design of spark plugs" by Professor R. V. Churchill, University of Michigan.

8. "Some applications of mathematics in physical chemistry" by L. O. Case, University of Michigan, by invitation.

9. "The imaginary in analytic geometry" by H. M. Ackley, Western State Teachers College.

10. "Functions whose ranges are simply ordered sets" by Professor K. W. Folley, College of the City of Detroit.

11. "A method for approximating real roots of equations by the principle of areas" by Professor T. R. Running, University of Michigan.

At the afternoon session of the Section the following program was given:

12. "An undergraduate course leading to the study of wave mechanics" by Professor W. D. Cairns, Oberlin College (Guest speaker).

13. "A special slide rule for a woolen mill problem" by Professor E. E. Ingalls, Albion College.

14. "Focal and intersector hypersurfaces from a projective standpoint" by W. M. Borgman, Jr., College of the City of Detroit.

LOUIS A. HOPKINS, *Secretary*

THE 1932 MEETING OF THE TEXAS SECTION

The 1932 meeting of the Texas Section of the Mathematical Association of America was held in Waggener Hall of the University of Texas, Austin, Texas, on Saturday, January 30, 1932. Local arrangements were made by a committee consisting of Professors H. J. Ettlinger, chairman, E. L. Dodd and R. G. Lubben.

The attendance was forty-four, including the following twenty-seven members of the Association: W. N. Barnes, P. M. Batchelder, L. W. Blau, A. A. Blumberg, L. M. Blumenthal, H. E. Bray, J. E. Burnam, E. L. Dodd, Nat Edmonson, Jr., H. J. Ettlinger, G. C. Evans, G. W. Evans, A. E. Finlay, L. R. Ford, C. A. Gilley, H. H. Halperin, J. W. Harrell, Goldie P. Horton, R. G. Lubben, R. L. Moore, C. A. Murray, Brother Norbert, P. K. Rees, W. A. Rees, B. P. Reinsch, C. R. Sherer, F. W. Sparks.

At the business meeting Professor C. R. Sherer of Texas Christian University was elected chairman of the Section for the coming year and W. A. Rees of the Houston Junior College was chosen vice-chairman for the same period.

The entertainment features of the meeting consisted of a sight-seeing trip about Austin conducted by members of the University of Texas mathematics department immediately following the afternoon session; and a dinner, complimentary to the members of the Texas Section and their guests, given by the University of Texas. The principal speaker at the dinner was Dr. H. Y. Benedict, President of the University.

The following program was presented:

1. "The isometric circle" by Professor L. R. Ford, The Rice Institute.
2. "Relations between projective geometry and trigonometry" by Professor M. B. Porter, University of Texas, by invitation.
3. "The postulate of continuity in dynamics" by Dr. C. H. Dix, The Rice Institute, by invitation.
4. "The cooling problem for spherical regions" by Dr. W. M. Rust, The Rice Institute, by invitation.
5. "Some recent trends and applications of statistics" by Professor E. L. Dodd, University of Texas.
6. "Simplifications for high school mathematics" by G. W. Evans.

Abstracts of these papers follow:

1. This paper gave a history of the isometric circle with some of the major results achieved by its use, and indicated some open problems together with possible avenues to their solution.

2. This paper discussed the place of projective geometry in undergraduate teaching together with some application to trigonometry. The straight line equation was interpreted as a projective transformation in one dimension, and the fundamental formulae of analytic trigonometry were derived.

3. In all classical theories dynamical quantities have been assumed to vary

over a continuous range of values. It is plain that a mathematical continuum corresponds to no direct observation and so enters the theory for reasons other than observational. In studying this problem the author has been led to deny the postulate and investigate any consequences of this denial. In place of Hamilton's principle consider

$$\delta \sum_{i=1}^{n} L(x(t_i),\ [x(t_i)]_t,\ t_i)\Delta_+ t_i = 0,$$

where $[x(t_i)]_t$ is the velocity $\{x(t_{i+1})-x(t_i)\}/\{t_{i+1}-t_i\}$ and $\Delta_+ t_i = t_{i+1}-t_i$. L is the Lagrangian function and δ indicates the first variation of the sum. We here postulate a finite sequence of instants $t_1, t_2, \cdots, t_{n+1}$. The result is the analog of the Lagrangian equation

$$\frac{\partial L}{\partial x} - \left[\frac{\partial L}{\partial [x]_t}\right] = 0,$$

where

$$[f(t_i)]_t = \{f(t_i) - f(t_{i-1})\}/\{t_{i+1} - t_i\}.$$

The transformation given by

$$p - \frac{\partial L}{\partial [x]_t} = 0$$

$$H(x,\ p,\ t_i) = [x]_t p - L(x,\ [x]_t,\ t_i)$$

leads to the Hamiltonian equations

$$[p]_t = -\frac{\partial H}{\partial x}, \qquad [x]_t = \frac{\partial H}{\partial p}.$$

It will be noticed that partial derivatives occur in spite of the discontinuity in the time, but these enter when we select from a continuum of possible $x(t_i)$'s the one, for each value of t_i, that makes the first variation of the sum zero. At this point it should be mentioned that in quantum mechanics the possible values of p and q are a continuum; but it cannot be said that the transitions are continuous. In conclusion it may be stated that the postulate of an exactly given sequence of instants can no more be placed on an observational basis than the postulate of a continuum. The discontinuous time furnishes a new parameter but at the same time introduces difficulties when we use coordinates other than rectangular ones since the square of the velocity as it appears in L is no longer in the simple form $[x]_t^2 + [y]_t^2$.

4. This paper treats the solution of the cooling problem for spherical regions by a slight modification of the methods used for linear regions in another paper.[1]

[1] *Integral equations and the cooling problem for several media*, American Journal of Mathematics, vol. 54 (1932), p. 190.

A general uniqueness theorem is established and a solution satisfying this theorem is constructed by use of the fundamental solution

$$\frac{1}{r(t-t')^{1/2}}\, e^{-a^2(r-r')^2/4(t-t')}$$

of the differential equation for the flow of heat in spherical regions. This solution is in terms of the solutions of a set of integral equations. It is shown that these integral equations can be solved by constructing a set of Volterra integral equations with bounded kernels, whose solutions are shown to be absolutely continuous and to have derivatives which satisfy the original set of integral equations.

5. As the title indicates, this paper consisted of a discussion of some late developments in the field of statistics. Particular attention was paid to applications in the biological sciences.

6. This paper discussed simple and straightforward methods already in use in some schools for arithmetic and algebra and proposed methods in geometry. These methods do not represent innovations, but a return to the ancient ways, with no omission of essential topics and with emphasis on the connection between "great ideas and ordinary thoughts," on generality and on self-criticism.

NAT EDMONSON, JR., *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE KANSAS SECTION

The eighteenth annual meeting of the Kansas Section of the Mathematical Association of America was held in the High School Building, Topeka, Saturday, February 13, 1932. The morning session was a joint meeting with the Kansas Association of Mathematics Teachers. The two organizations held separate meetings in the afternoon. Professor J. J. Wheeler, chairman of the Section, presided at both the morning and afternoon sessions.

Sixty-four persons were in attendance, including the following twenty-six members of the Association: (Note: This was a fifty per cent attendance of the Kansas membership.) C. H. Ashton, R. W. Babcock, Wealthy Babcock, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, W. H. Hill, A. S. Householder, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, Thirza A. Mossman, Arthur Ollivier, O. J. Peterson, A. W. Philips, P. S. Pretz, B. L. Remick, J. A. G. Shirk, G. W. Smith, W. T. Stratton, A. E. White, Ferna E. Wrestler.

At the business session, the following officers were elected for the coming year: Chairman, Professor O. J. Peterson, State Teachers College, Emporia; Vice-Chairman, Professor W. A. Harshbarger, Washburn College; Secretary-Treasurer, Lucy T. Dougherty, Junior College, Kansas City.

Between sessions the two Associations enjoyed a most delightful luncheon,

served in the Faculty dining room of the palatial new building of the Topeka High School.

At the joint session the following program was presented:

1. "Every day class-room problems" by Amy Irene Moore, State Teachers College, Hays.

2. "Common statistical errors" by Professor Dinsmore Alter, Department of Astronomy, University of Kansas.

3. "Orientation in mathematics" by Professor O. J. Peterson, State Teachers College, Emporia.

At the session in the afternoon, the following papers were presented:

1. "Singular points of polar tangent curves" by Professor W. T. Stratton, State College, Manhattan.

2. "Recent advances in mathematical statistics" by Professor J. A. G. Shirk, State Teachers College, Pittsburg.

3. "On contact of curves in a four-dimensional space" by B. C. Moore, University of Kansas, by invitation.

Abstracts of these papers follow:

1. In this discussion of the general polar tangent curves $\rho = a \tan(p\theta/q) + k$ where p and q are integers and a and k are any constants, Professor Stratton showed that the curves are algebraic and of order $2(p+q)$ if q is odd and of order $(p+q)$ if q is even. He next determined the axes of symmetry, the polar period and the conditions for maxima and minima, and then pointed out certain properties common to all of the curves. The greater part of the paper was devoted to the development of the number of singular points other than the origin. The number of double points can readily be expressed in terms of p and q.

2. Professor Shirk showed large graphs of various distributions, each one being fitted with the Pearsonian equation best adapted. The distributions of the mean and the standard deviation were shown, with a discussion of the wrong use of the probable error of the standard deviation as quite frequently used. Graphs of correlated varities were also shown, together with graphs of the distribution of the correlation coefficient, which is generally quite different from the normal distribution. Fisher's methods of testing the significance of the standard deviation and of the correlation coefficient were presented. Fisher's method of curve fitting by the method of "Maximum likelihood" was illustrated, and its greater efficiency over the "Method of moments" was indicated.

3. Mr. Moore showed that in four-dimensional space two curves which have contact of order n determine a principal tangent plane. In general the curves do not determine a principal line or a principal point. It was shown that the order of contact of the projections of the two curves from a line on the common osculating plane may be made, by proper choices of the line, as high as $n+4$. The method of proof is that devised by E. B. Stouffer in proving analogous theorems in three-dimensional space.

LUCY T. DOUGHERTY, *Secretary*

ON THE PROBLEM OF RUNS

By J. V. USPENSKY, Stanford University

Let us consider a series of n trials independent in regard to a certain event E, the probability of this event being p in each trial. If, in the course of the trials, the event E occurs at least r times in succession, we say that there is a *run of r* successes. For instance, if, tossing a coin 1000 times, heads occur 20 or more times in succession, we have a run of 20 heads in a series of 1000 tosses. The problem of finding the probability that there should be a run of r successes in a series of n trials is an old one propounded and solved by De Moivre. The same problem has been discussed by many other writers on probability, but none of them, so far as I am aware, ever gave an expression for the required probability adapted to the particularly important case of a very long series of trials.

It is the purpose of this note to establish approximate formulae adapted to the case of a long series of trials.

1. De Moivre reduced the problem to an equation in finite differences. Let y_n be the probability of a run of r in n trials. It is easy then to establish the following equation in finite differences:

$$y_{n+1} = y_n + (1 - y_{n-r})p^r q$$

where, as usual, $q = 1 - p$ is the probability of a failure. Since evidently

$$y_0 = y_1 = \cdots = y_{r-1} = 0,\ y_r = p^r,$$

the preceding equation gives a means to determine successively y_{r+1}, y_{r+2}, y_{r+3} and in general y_n for every $n > r$. One can even establish a general expression for y_n, but this expression has little practical value if n is a large number.

It is more convenient to consider not y_n but the complementary probability $z_n = 1 - y_n$ determined by the equation

$$z_{n+1} - z_n + qp^r z_{n-r} = 0 \tag{1}$$

together with the initial conditions

$$z_0 = z_1 = \cdots = z_{r-1} = 1;\ z_r = 1 - p^r. \tag{2}$$

Using (1) and (2) one easily finds the following expression for the generating function of probabilities $z_0, z_1, z_2, \cdots$:

$$\phi(\xi) = \frac{1 - p^r\xi^r}{1 - \xi + qp^r\xi^{r+1}} = \sum_{n=0}^{\infty} z_n\xi^n. \tag{3}$$

The natural way to obtain an appropriate expression of z_n is to resolve the rational function in the left member of (3) into simple fractions corresponding to various roots of the denominator and expand those fractions in power series of ξ. However, to attain definite conclusions following this method, we must first seek information concerning roots of the equation

$$1 - \xi + qp^r\xi^{r+1} = 0.$$

2. Let

$$f(\xi) = \xi - 1 - \alpha\xi^{r+1},$$

where

$$\alpha = p^r(1 - p).$$

When p varies from 0 to 1 the maximum of $p^r(1-p)$ is attained for $p=r/(r+1)$ and is $r^r/(r+1)^{r+1}$, so that $\alpha \leqq r^r/(r+1)^{r+1}$ in all cases. To deal with the most interesting case, we shall assume

$$p < r/(r + 1) \tag{4}$$

which involves

$$\alpha < r^r/(r + 1)^{r+1}$$

and we leave it to the reader to find out how the following discussion should be modified if $p \geqq r/(r+1)$.

When ξ starts to increase from 0 the function $f(\xi)$ steadily increases and attains a positive maximum for $\xi = \xi_0$ where

$$(r + 1)\alpha\xi_0{}^r = 1,$$

after which $f(\xi)$ decreases steadily to negative infinity. Hence there are two positive roots of the equation $f(\xi) = 0$: ξ_1 which is less than $(r+1)/r$ and another root greater than this number. This root is $1/p$ if the condition (4) is fulfilled.

The remaining roots are all imaginary if r is *odd* and there is one negative root among them if r is *even*.

Now we shall prove that the absolute value of every imaginary or negative root is $>1/p$. For let ρ be the absolute value of any such root. We have first

$$f(\rho) = \rho - 1 - \alpha\rho^{r+1} < 0$$

so that ρ belongs either to the interval $(0, \xi_1)$ or to the interval $(1/p, \infty)$, and if we can show that $\rho > \xi_0$ then ρ can be only $>1/p$. If the root we consider is negative, ρ satisfies the equation

$$F(\rho) = 1 + \rho - \alpha\rho^{r+1} = 0$$

and since $F(\rho)$ increases till a positive maximum for $\rho = \xi_0$ is reached, and then decreases, the root of $F(\rho) = 0$ is necessarily $>\xi_0$. If $\xi = \rho e^{i\theta}$ is an imaginary root of $f(\xi) = 0$ we have, equating imaginary parts,

$$\alpha\rho^r \frac{\sin (r + 1)\theta}{\sin \theta} = 1. \tag{5}$$

But whatever θ may be

$$\left| \frac{\sin (r + 1)\theta}{\sin \theta} \right| \leqq r + 1,$$

the equality sign being excluded if $\sin\theta \neq 0$. Hence

$$(r+1)\alpha\rho^r > 1$$

which implies $\rho > \xi_0$. The statement is thus completely proved.

3. The equation

$$\xi - 1 - \alpha\xi^{r+1} = 0$$

can be exhibited in the form

$$\frac{1}{\xi} + \alpha\xi^r = 1$$

Substituting here $\xi = \rho e^{i\theta}$ and again equating imaginary parts, we get

$$\alpha\rho^{r+1}\sin r\theta = \sin\theta,$$

and, combining this with (5),

$$\rho = \frac{\sin(r+1)\theta}{\sin r\theta}, \qquad \alpha = \frac{(\sin r\theta)^r \sin\theta}{[\sin(r+1)\theta]^{r+1}}.$$

If the imaginary part of ξ is positive the argument θ is contained between 0 and π. In this case it cannot be less than $\pi/(r+1)$ nor greater than $\pi - \pi/(r+1)$. For, if $0 < \theta < \pi/(r+1)$,

$$\frac{\sin r\theta}{r\theta} > \frac{\sin(r+1)\theta}{(r+1)\theta},$$

or

$$\frac{\sin r\theta}{\sin(r+1)\theta} > \frac{r}{r+1}.$$

At the same time

$$\frac{\sin\theta}{\sin(r+1)\theta} > \frac{1}{r+1},$$

and hence

$$\alpha = \left\{\frac{\sin r\theta}{\sin(r+1)\theta}\right\}^r \frac{\sin\theta}{\sin(r+1)\theta} > \frac{r^r}{(r+1)^{r+1}}$$

which is impossible. That θ cannot be greater than $\pi - \pi/(r+1)$ follows simply because in this case $\sin(r+1)\theta$ and $\sin r\theta$ would be of opposite signs and ρ would be negative.

As $\pi/(r+1) \leqq \theta \leqq \pi - \pi/(r+1)$, we have

$$\rho\sin\theta \geqq \rho\sin\frac{\pi}{r+1}.$$

On the other hand $\sin x > 2x/\pi$ if $0 < x < \pi/2$, and $\rho > 1/p$. Hence

$$\rho \sin\theta > \frac{2}{(r+1)p}.$$

Thus imaginary parts of all complex roots have the same lower bound

$$\frac{2}{(r+1)p}$$

of their absolute values.

4. Denoting the roots of the equation $f(\xi) = 0$ by $\xi_k (k = 1, 2, 3, \cdots, r+1)$ we have

$$\phi(\xi) = \sum_{k=1}^{r+1} \frac{1 - p\xi_k}{(1-p)\xi_k(r+1-r\xi_k)} \left(1 - \frac{\xi}{\xi_k}\right)^{-1}.$$

Hence, expanding each term into power series of ξ and collecting coefficients of ξ^n we find

$$z_n = \sum_{k=1}^{r+1} \frac{1 - p\xi_k}{(1-p)\xi_k} \cdot \frac{\xi_k^{-n}}{r+1-r\xi_k}.$$

For every imaginary root we have

$$\left| \frac{(1-p\xi_k)\xi_k^{-n}}{(1-p)\xi_k(r+1-r\xi_k)} \right| < \frac{r+1}{r(1-p)} p^{n+2}$$

since

$$|\xi_k|^{-1} < p; \left| \frac{1}{\xi_k} - p \right| < 2p, \quad \frac{1}{|r+1-r\xi_k|} < \frac{(r+1)p}{2r}.$$

If r is *odd* there are $r-1$ imaginary roots and the part in the expression of z_n due to them in absolute value is less than

$$\frac{(r+1)(r-1)}{r(1-p)} p^{n+2} < \frac{r}{1-p} p^{n+2}.$$

The term corresponding to the root $1/p$ vanishes, so that finally

$$z_n = \frac{1 - p\xi_1}{(1-p)\xi_1} \cdot \frac{\xi_1^{-n}}{r+1-r\xi_1} + \theta \frac{r}{1-p} p^{n+2}$$

where $|\theta| < 1$ and ξ_1 denotes the least positive root of the equation

$$\xi = 1 + \alpha\xi^{r+1}.$$

If r is *even* there is one negative root. The part of z_n corresponding to this root is less than

$$\frac{2p^{n+2}}{(1-p)r}.$$

The whole contribution due to imaginary and negative roots in absolute value is less than

$$\frac{r^2-r}{r(1-p)}\,p^{n+2} < \frac{r}{1-p}\,p^{n+2}$$

Thus no matter whether r is odd or even, we have

$$z_n = \frac{1-p\xi_1}{(1-p)\xi_1}\cdot\frac{\xi_1^{-n}}{r+1-r\xi_1} + \theta\frac{r}{1-p}\,p^{n+2};\ -1<\theta<1. \tag{6}$$

This is the required expression for z_n, excellently adapted to the case of a large n since then the remainder term involving θ is completely negligible in comparison with the first principal term.

The root ξ_1 can be found either by direct solution of the trinomial equation following Gauss's method, or by application of Lagrange's series. Applying Lagrange's series we have

$$\xi_1 = 1+\alpha+\sum_{l=2}^{\infty}\frac{(lr+2)(lr+3)\cdots(lr+l)}{l!}\alpha^l$$

$$\log\xi_1 = \alpha+\sum_{l=2}^{\infty}\frac{(lr+1)(lr+2)\cdots(lr+l-1)}{l!}\alpha^l$$

both series being convergent if $|\alpha| < r^r/(r+1)^{r+1}$, and this condition is satisfied.

5. Let us apply the approximate formula (6) to the case $p=q=\frac{1}{2}$ and $r=10$. Using Lagrange's series we find

$$\xi_1 = 1.0004908$$

and

$$z_n = 1.003937\times(1.0004908)^{-n} + 5\theta/2^n.$$

Hence for $n=100$, 1000, 10000 respectively

$$z_n = 0.9559;\ 0.6148;\ 0.0074$$

so that, for instance, the probabilities of a run of at least 10 heads in 100, 1000, 10000 throws of a coin are, respectively

$$0.0441;\ 0.3852;\ 0.9926.$$

Thus in 10,000 throws it is quite likely that heads would turn up 10 or more times in succession.

In general for a given r and increasing n the probability y_n tends to 1, so that in a very long series of trials runs of any length are extremely likely to occur, a conclusion which at first sight might seem paradoxical.

THE PROJECTIVE THEORY OF ORTHOPOLES

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Introduction

The term "orthopole of a line with respect to a given triangle" was first used by Neuberg to designate a point constructed as follows:[1] Perpendiculars are drawn from the vertices of the given triangle $A_1A_2A_3$ to the given line l, cutting l in B_1, B_2, B_3, respectively. Then are drawn lines B_1C_1, B_2C_2, B_3C_3, perpendicular respectively to A_2A_3, A_3A_1, A_1A_2. These three perpendiculars are concurrent on a point M which is the orthopole of l with respect to the given triangle. Neuberg showed that while the line l gave rise to a unique orthopole M, yet M was equally well determined by each of two other lines. The three lines that gave rise to a common orthopole he designated as "associated" lines. He also proved that when the orthopole varied on a line, the three associated lines varied on a line parabola.

It is the purpose of this paper to study this (1, 1) correspondence between lines and line parabolas, to deduce therefrom the (1,3) correspondence between orthopoles and their associated lines, and then to prove by means of projective geometry some theorems concerning orthopoles. The theorems, with the exception of 1 and its Corollary, 16, 17, and the construction for finding the lines associated with a given one, have all been given before. They may be found in the works of Neuberg,[2] Cwojdzinski,[3] Goormaghtigh,[4] Ramler.[5] In the proofs as given by these authors, the methods of synthetic and analytical geometry have been used, various types of coordinate systems being employed.

Since the (1,3) correspondence will be defined by means of a net of line conics, some preliminary theorems concerning such nets will be cited.[6]

Theorem A. In a net of line conics, the point pairs that are the centers of the pencils that constitute the degenerate conics of the net, lie on a cubic,[7] which, in the case of a base line, degnerates into that line, l, and a residual conic, C^2.

The following theorems refer to nets of line conics having a base line.

Theorem B. There is a (1,1) correspondence between the points on l and those on C^2, each pair of corresponding points forming a degenerate conic. In

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[1] *Sur une transformation des figures*, Nouv. Corr. Math., vol. 4, p. 379, 1878.

[2] *Die Verwandtschaft zwischen einer Geraden und ihrem Lotpunkt in Bezug auf ein Dreieck*, Archiv der Mathematik und Physik, vol. 3, p. 89, 1902.

[3] *Der Lotpunkt, ein neuer merkwürdiger Punkt des Dreiecks*, Archiv der Mathematik und Physik, vol. 1, p. 175, 1901.

[4] *The Orthopole*, Tohoku Mathematical Journal, 1927, p. 78.

[5] *The orthopole loci of some one-parameter systems of lines referred to a fixed triangle*, American Mathematical Monthly, vol. 37, p. 130. March 1930.

[6] Proofs of these theorems may be found in the original copy of the writer's dissertation on the subject in the Johns Hopkins University library.

[7] Reye, *Geometry of Position* (1898), p. 231.

this correspondence, the points common to l and C^2 correspond doubly, i.e., to each considered as a point on l corresponds the other considered as a point on C^2, and to each considered as a point on C^2 corresponds the other considered as a point on l.

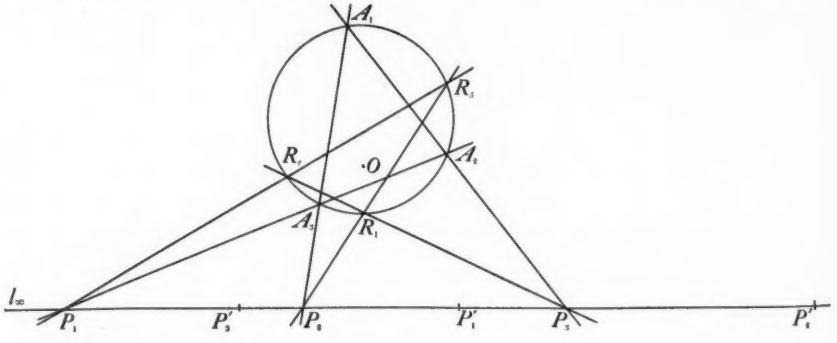
Theorem C. If the centers of the pencils that constitute degenerate conics of the net be joined to any point P on C^2, these lines are conjugate pairs in an involution of lines on P.

Theorem D. If a line l and a conic C^2 be chosen arbitrarily, the net having these for the locus of the centers of the pencils that form the degenerate conics is fixed by assigning arbitrarily one point pair—one point on l, the other on C^2—as one degenerate line conic of the net, the points common to l and C^2 being the centers of a second degenerate conic.

Theorem E. If 1, 2, 3 are three points on l which form with the three vertices of a triangle inscribed in C^2 three degenerate conics, and if 1′, 2′, 3′ are the points in which the sides of the triangle opposite 1, 2, 3, respectively, cut l, then the three pairs 1,1′; 2,2′; 3,3′ are conjugate pairs of an involution.

Basis for Choice of Net and Determining Elements of the Correspondence

Neuberg proved that as the orthopole varies on the side A_1A_2 of the base triangle, the three associated lines vary on the degenerate line parabola having as centers of the two pencils of lines R_3 and P_3'; on A_2A_3, centers R_1, P_1'; on A_1A_3, centers R_2, P_2'; where R_1, R_2, R_3 are points on the circumcircle diametrically opposite to A_1, A_2, A_3; and P_1', P_2', P_3' are conjugates in the absolute involution on l_∞ with respect to the points P_1, P_2, P_3, these being the points of intersection of l_∞ and the sides of the triangle respectively opposite to A_1, A_2, A_3.



The triangles $A_1A_2A_3$ and $R_1R_2R_3$, being perspective from center O of the circle, are also perspective from l_∞. It is readily seen that the preceding is an example of Theorem E, the involution of points on l being the absolute involution.

Neuberg also proved that as a line varies on a point P on the circumcircle

of the base triangle, its orthopole varies on the Simson line of that point with respect to the base triangle. Since l_∞ cuts the circle in the circular points, I and J, its orthopole is the intersection of the Simson lines of I and J. These lines, however, coincide with l_∞. Thus, any point on l_∞ is the orthopole of l_∞. Also, the two lines associated with l_∞ pass one through I and the other through J, the two intersecting at some point K on the circle. It follows that as a point varies on l_∞, the associated lines vary on the degenerate parabola IJ.

In selecting the elements that determine the (1,3) correspondence that will be defined later, the choice is so made that to the three sides of the base triangle $A_1A_2A_3$ will correspond the parabolas as given by Neuberg, and that to l_∞ will correspond IJ. Since four independent pairs of the (1,3) correspondence give rise to the same four pairs in the Neuberg definition, the two definitions must be equivalent.

Determination of the Net and the Correspondence

The net of line conics to be used is set up as follows:

(a) Let l_∞ be chosen as base line. Then all the conics are parabolas.

(b) The absolute involution on l_∞ is fixed by its double points I and J. Let IJ be a degenerate conic of the net. Then, by Theorem B, C^2 must pass through I and J, and consequently is a circle.

(c) Let $R_1R_2R_3$ be any triangle inscribed in this circle, and let R_1R_2, R_2R_3, R_3R_1 cut l_∞ in P_3, P_2, P_1, respectively. Also, let P_1P_1', P_2P_2', P_3P_3' be conjugate pairs in the absolute involution. The last needed element to determine the net, according to Theorem D, is chosen as degenerate conic R_1P_1'. But then, by Theorem E, R_2P_2', R_3P_3' are also degenerate conics.

The base triangle $A_1A_2A_3$, with respect to which the orthopoles of lines of the plane are defined is chosen as the triangle perspective with $R_1R_2R_3$ from center O of the circle. The triangles will also be perspective from l_∞.

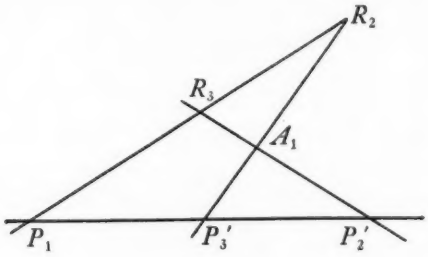
Consider now a projective correspondence between the lines of the plane and the ∞^2 line parabolas of the above linear system. In the (1,1) correspondence, to l_1 and l_2, let correspond parabolas π_1 and π_2, respectively. Then to the intersection of l_1 and l_2 must correspond the range determined by π_1 and π_2. A (1,3) correspondence between points and lines may now be defined. To the point of intersection of l_1 and l_2 correspond the three lines, exclusive of l_∞, common to all parabolas of the range determined by π_1 and π_2. In order that the correspondence be that of orthopoles and associated lines, the determining pairs of homologous elements are chosen as follows: To A_1A_2, let correspond the degenerate parabola R_3P_3'; to A_2A_3, R_1P_1'; to A_1A_3, R_2P_2'; and to l_∞, IJ.

Further Correspondences Derived from the Defining Elements

Since A_1 is common to A_1A_2 and A_1A_3, to A_1 corresponds the range containing degenerate parabolas R_3P_3' and R_2P_2'. The three lines that correspond to A_1 are R_2R_3, R_2P_3', R_3P_2', the two latter intersecting at A_1 on the circle. Also, R_2R_3 cuts l_∞ in P_1. It follows that A_1P_1 is the third degenerate parabola of the range.

In terms of orthopoles, A_1 is the orthopole of R_2R_3, R_2P_3', R_3P_2'; and similarly, A_2 is orthopole of R_1R_3, R_1P_3', R_3P_1'; and A_3, of R_1R_2, R_1P_2', R_2P_1'.

Since P_1 is common to A_2A_3 and l_∞, to P_1 corresponds the range containing degenerate parabolas R_1P_1' and IJ. Thus, P_1 is orthopole of R_1I, R_1J, and l_∞; i.e., l_∞ is a double tangent of parabolas of the range. If in the figure below, two adjacent sides of the quadrilateral become coincident, the points of contact of the inscribed parabolas with those two sides approach each other and also the point of intersection of the two sides. When the two sides coincide, the parabolas will be tangent at that point which forms with the remaining vertex a degenerate parabola. It follows that to P_1 corresponds the range containing degenerate parabolas R_1P_1' and IJ, P_1' being the point of contact; to P_2, R_2P_2' and IJ, P_2' being the point of contact; and to P_3, R_3P_3' and IJ, P_3' being the point of contact.



The point pairs that form the centers of the pencils which are the degenerate conics of the net lie one on the circle and the other on l_∞. Also the three lines that correspond to a given point are always the sides, exclusive of l_∞, of the quadrilateral in which parabolas of the range are inscribed. It follows that the three associated lines always intersect by pairs in three points on the circle. These sets, of three points each, constitute a linear series, a g_3^2, on the circle. It is known that to every g_n^r on a plane algebraic curve, there corresponds a curve of order n, in a space S_r, in (1,1) correspondence with the given curve, on which the image of the g_n^r is the series of ∞^r points cut out by the hyperplanes of that space. Therefore the image of the g_3^2 on the circle is the series of ∞^2 points cut out on a plane cubic curve by the ∞^2 lines of the plane of that cubic, i.e., a g_3^2 on a plane cubic curve.

From this we deduce that there are three types of sets of points on the circle:

(1) Three distinct points, and thus three distinct associated lines—corresponding to the three points in which a generic line cuts the cubic.

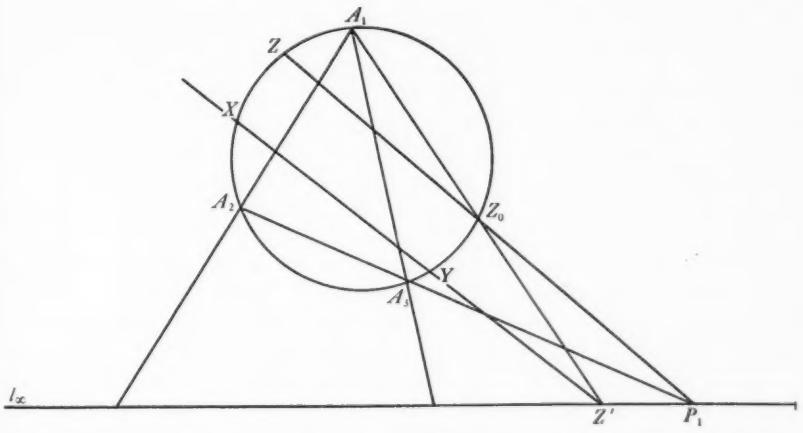
(2) Two of the points coincident, and thus two of the associated lines coincident, the third tangent to the circle at the double point—corresponding to the points in which a tangent line cuts the cubic.

(3) Sets having two points a neutral pair of the g_3^2, in common, and a third

variable point—corresponding to the points in which a line through the double point of the cubic cuts it. It was shown previously that I and J belong to more than one set. They must therefore constitute this neutral pair. For sets of this type, the three associated lines are l_∞ and two others, one on I, the other on J, giving rise to ranges having only two degenerate conics, with a fixed point of contact on l_∞.

Construction

Problem. Given the line XY and the base triangle $A_1A_2A_3$, to construct the two lines XZ and YZ, associated with XY.



Let XY cut l_∞ in Z'. Join A_1 to Z', letting A_1Z' cut the circle again in Z_0. Join Z_0 to P_1, and let the join cut the circle at Z. Then ZX and ZY are the required lines.

Proof. Let π be the projectivity between points on the circle and points on l_∞ that together form degenerate conics of the net (Theorem B). Then

$$\pi(IJA_1A_2A_3R_1R_2R_3) = JIP_1P_2P_3P_1'P_2'P_3'.$$

Let K be any point on the circle. Then

$$KI, KJ, KA_1, \cdots \bar{\wedge} KJ, KI, KP_1, \cdots.$$

Since this latter projectivity interchanges the pair KI, KJ, it is an involution. Now, for the pair KA_1, KP_1, A_1P_1 is a degenerate conic. It follows from the involution, that the other points, M, in which KA_1 cuts l_∞, and N, in which KP_1 cuts the circle must also constitute a degenerate conic. And finally, the line A_1N will cut l_∞ in the point L that forms with K a degenerate conic. There is thus an involution of points on the circle, center L.

In the construction as given, $K \equiv A_1$. Then $L \equiv P_1$. Let I_1 be the involution

of points on the circle, center P_1; and I_2, the involution of lines on A_1 passing through conjugate pairs of I_1 on the circle. Then $I_1(Z) = Z_0$ and $I_2(A_1Z_0) = A_1Z$. Therefore, Z and Z' form a degenerate conic of the range inscribed in XY, XZ, YZ, and l_∞. Thus, the lines XZ and YZ are associated with XY.

Some Theorems on Orthopoles

Theorem 1. Every point X on l_∞ is the orthopole of three lines, IJ, IK, JK, where K is the point on the circle that forms with the conjugate of X in the absolute involution a degenerate conic.

It was stated previously that to points on l_∞ correspond ranges having only two degenerate parabolas, one being IJ, the other consisting of a point K on the circle and a fixed point of contact, K', on l_∞, for all parabolas of the range. There is thus a (1,1) correspondence between the orthopole X and the point K on the circle. Let this correspondence be π. Let the correspondence between the point K on the circle and K' on l_∞ that forms with it a degenerate conic be σ. Finally, let the correspondence between X and K' be ω.

Now $\pi(X) = K$ and $\sigma(K) = K'$. But $\omega(X) = K'$. Therefore, $\pi\sigma = \omega$. It is known that $\pi(P_1P_2P_3) = R_1R_2R_3$ and $\sigma(R_1R_2R_3) = P_1'P_2'P_3'$. Thus $\omega(P_1P_2P_3) = P_1'P_2'P_3'$. Therefore, ω must coincide with the absolute involution and thus the theorem is proved.

Remark. Since π transforms three real points on l_∞ into three real points on the circle, the projectivity π must be real.

Corollary. Each of the circular points I and J is the orthopole of l_∞ taken twice and the tangent OJ and OI, respectively, at the other circular point, where O is the center of the circle.

Since $\pi\sigma = \omega$, $\pi = \omega\sigma^{-1}$. Now $\omega(IJ) = IJ$, and $\sigma^{-1}(IJ) = JI$. Therefore $\pi(IJ) = JI$. It follows that I is the orthopole of the lines determined by the set IJJ, while J is the orthopole of the lines determined by the set IIJ. The three associated lines are then l_∞ counted twice and the tangent to the circle at the double point of the set which in each case is the circular point other than the orthopole.

Theorem 2. The orthocenter of the base triangle is the orthopole of its three sides.

Since $\omega(P_1'P_2'P_3') = P_1P_2P_3$ and $\sigma^{-1}(P_1P_2P_3) = A_1A_2A_3$, $\pi(P_1'P_2'P_3') = A_1A_2A_3$. Thus P_1' has corresponding to it the range having IJ and A_1P_1 as degenerate parabolas, P_1 being the fixed point of contact. Similarly, to P_2' corresponds the range having IJ and A_2P_2 as degenerate parabolas, with P_2 the fixed point of contact. And finally, to P_3' corresponds the range having IJ and A_3P_3 as degenerate parabolas, with P_3 as fixed point of contact.

Since the ranges that correspond to the points A_1 and P_1' have in common the degenerate parabola A_1P_1, that parabola must correspond to the line A_1P_1'. Similarly, parabolas A_2P_2 and A_3P_3 correspond respectively to lines A_2P_2' and A_3P_3'. Furthermore, since the three lines A_1P_1', A_2P_2', and A_3P_3' are concurrent in H, the orthocenter of the triangle $A_1A_2A_3$, to H must correspond the range

inscribed in $A_1A_2A_3$ and l_∞. Or, in terms of orthopoles, H is the orthopole of the three sides of the triangle $A_1A_2A_3$.

Theorem 3. To every degenerate conic of the net corresponds the Simson line of a point on the circle with respect to the base triangle.

To every point P of the plane corresponds a range of parabolas. To the three degenerate parabolas of the range correspond three lines on P. As P varies in the plane, these lines envelope a curve of class three. It has already been shown that the three sides and altitudes of the base triangle, also l_∞, have degenerate parabolas corresponding to them. Furthermore, by Theorem 1, Corollary, $\pi(IJ) = JI$. This means that for the set IIJ, having l_∞ as double line, J is the point of contact and hence a point of the envelope. Similarly, I is a point of the envelope.

The envelope of the Simson lines with respect to a given triangle of points on its circumcircle is known to be a three-cusped hypocycloid, having l_∞ as double tangent at the circular points. Each of the seven lines above mentioned is a Simson line with respect to the base triangle. Thus the Simson line envelope and the envelope first considered have in common six tangents, one double tangent and its two points of contact. Dually, there are two cubics having in common six points, one double point, and common tangents at that double point—making a total of twelve intersections. The two cubics, having in common more than nine points, must coincide. Dually, the two envelopes under consideration must be the same. Therefore, every line which has a degenerate parabola corresponding to it is a Simson line, and conversely.

Theorem 4. A Simson line is always perpendicular to the double line of the degenerate parabola to which it corresponds.

Let AB be the double line of the degenerate parabola that corresponds to the Simson line, s, A being on l_∞, B on the circle. Let AA' be a conjugate pair in the absolute involution. By Theorem 1, to A' corresponds the range having IJ and AB as degenerate parabolas. Therefore, all parabolas of the range correspond to lines on A'. Then the Simson line, s, to which AB corresponds must pass through A', and consequently is perpendicular to line AB.

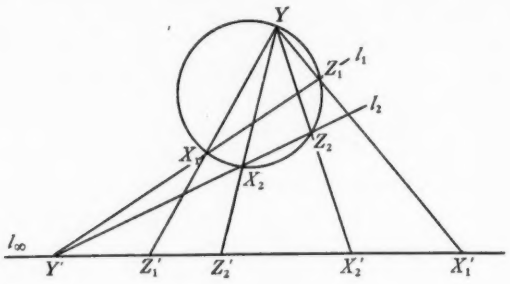
Theorem 5. The orthopole of an altitude of the base triangle is the foot of that altitude.[1]

Theorem 6. When a line l moves parallel to itself, its orthopole moves on a line perpendicular to l, and the two associated lines pass through a fixed point on the circumference of the circle.

Let P be the point to which correspond the lines X_1Z_1, YX_1, YZ_1. To the degenerate parabola YY' corresponds the Simson line perpendicular to the line YY', and the Simson line must also pass through P. For any other position of the line on Y', the range that corresponds to the new orthopole will always contain the degenerate parabola YY'. It follows that the orthopole of any line on Y' lies on the Simson line perpendicular to the line YY'. Also, the two lines

[1] Proofs of Theorems 5, 7, 9 may be found in original copy of writer's dissertation, loc. cit.

associated with the given line will always pass through that point on the circle that forms with Y' a degenerate parabola.



Theorem 7. The orthopole of a line perpendicular to a side of the base triangle is the foot of that perpendicular.

Theorem 8. As a line l varies on a curve of class m, its orthopole M varies on a curve of order $2m$.

As M moves on a line k, l envelopes a line parabola π. As M moves on a curve C, of order x, l envelopes a curve S of class m. To the common points of k and C, will correspond the lines common to π and S. These are $2m$ in number. Therefore the order x of C is $2m$.

Corollary. As a line m varies in a pencil of lines on a point P, the orthopole M varies on a conic, an ellipse.

In this case, m equals one; the locus of M is thus a conic. It remains to show that this is an ellipse, i.e., cuts l_∞ in two imaginary points. Since P is not on l_∞, the only lines on P that can have orthopoles on l_∞ are PI and PJ. For P real, the other point of intersection, Q, of PI with the circle must be imaginary. The projectivity π between l_∞ and points on the circle, by the Remark, Theorem 1, is real. Therefore, the point R that forms a degenerate conic with Q is imaginary. To the range QIJ, then, corresponds the point R', the conjugate of R in the absolute involution. But R', also imaginary, is then the orthopole of QI or PI. Similarly, the orthopole of PJ is imaginary. Thus the conic which is the orthopole locus is an ellipse.

Theorem 9. When a line m varies in a pencil of lines on a point P on l_∞, the orthopole locus degenerates into the Simson line perpendicular to m, and the line l_∞.

Theorem 10. When a line varies on a point P on the circumcircle of the base triangle, the orthopole locus is the Simson line of the point P with respect to the base triangle taken twice.

The feet of the three perpendiculars from P to the sides of the base triangle are their respective orthopoles (Theorem 7). But these points determine the so-called Simson line of the point P with respect to the base triangle. By Theorem 6, there are always two lines on P that give rise to the same orthopole. Thus the Simson line must be traced out twice.

Corollary. The orthopole of a line cutting the circumcircle in points P and Q is the intersection of the Simson lines of P and Q.

Theorem 11. As a point M describes a curve of order n, the three associated lines of which M is orthopole envelope a curve of class $2n$.

As a line l varies on a point P, its orthopole describes a conic (Theorem 8, Corollary). This conic cuts the locus C of M in $2n$ points. To each of these points must correspond a line on P tangent to the envelope described by the lines as M varies on C. Therefore the class of the envelope is $2n$.

Theorem 12. As a line varies on the center O of the circumcircle of the base triangle, its orthopole varies on the nine-point circle of the base triangle.

The locus, a conic, must be a circle since it passes through the circular points I and J, the respective orthopoles of tangents OJ and OI (Theorem 1, Corollary). Also the feet of the three perpendiculars from O to the three sides of the base triangle are on the circle. Thus the required locus is the nine-point circle of the base triangle.

Theorem 13. As a line varies on the hypocycloid which is the Simson line envelope, the orthopole locus is a quartic, having a triple point at the orthocenter H, and passing through the feet of the three altitudes of the base triangle.

The hypocycloid being of class three, the orthopole locus must be of order six. This degenerates into l_∞ counted twice and a residual quartic. That l_∞ is a double line of the locus follows from the fact that l_∞ is a double tangent to the hypocycloid, and has as orthopole any point on l_∞. Since the three sides of the base triangle are tangent to the hypocycloid, and each has H as orthopole, H is a triple point of the quartic. Finally, the altitudes are Simson lines having their respective feet as orthopoles. Thus the quartic also passes through the feet of the altitudes.

Theorem 14. As a line m varies as tangent to the circumcircle of the base triangle, its orthopole varies on the hypocycloid envelope of the Simson lines.

The tangents to the hypocycloid are all Simson lines and thus each gives rise to a degenerate conic of the net. From points not on the hypocycloid, three distinct tangents may be drawn. The corresponding range will thus have three distinct degenerate parabolas, and the associated lines will be distinct. But for points on the hypocycloid, two of the tangents will be coincident and the third distinct. The corresponding ranges will have two coincident degenerate parabolas, and the set of points on the circle has a double point. Of the three associated lines, one will be tangent to the circle at that double point, and the other will be the line joining the double point to the remaining point, this line being counted twice. Thus, for all points on the hypocycloid, one of the three associated lines will be tangent to the circle. Therefore, as a line varies as tangent to the circumcircle, its orthopole varies on the hypocycloid.

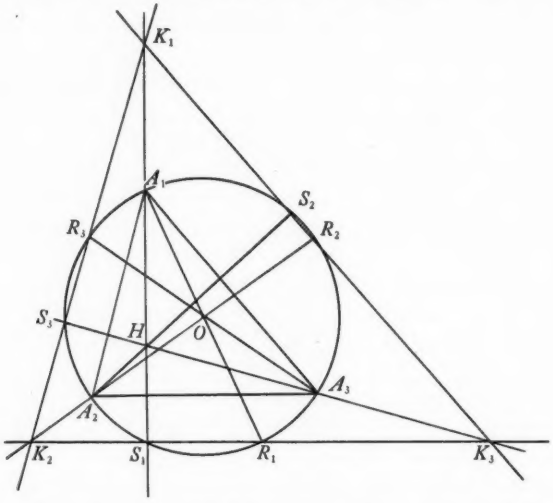
Corollary. If a line is tangent to the circumcircle, its orthopole is the point in which the Simson line of the point of contact touches the hypocycloid.

Theorem 15. Given any two sets of associated lines, the orthopole of either set with respect to the other as base triangle is the midpoint of the join of their orthocenters.

The following is a known property[1] of Simson lines: If the Simson lines of the points B_1, B_2, B_3, with respect to the triangle $A_1A_2A_3$ are concurrent on a point M, then the Simson lines of A_1, A_2, A_3, with respect to triangle $B_1B_2B_3$ are also concurrent on M, and M is midway between the orthocenters of the two triangles. From this property and the Corollary to Theorem 10, the above theorem obviously follows.

Theorem 16. As a point varies on the nine-point circle of the base triangle, its ortholines envelope a curve of class four, which degenerates into a three-cusped hypocycloid and the center of the circumcircle of the base triangle.

As the orthopole varies on a conic, the associated lines envelope a curve of class four. By Theorem 12, the center of the circumcircle is part of the desired locus. Consequently the residual portion is of class three.



Since one of the three associated lines is always on the center O, and thus a diameter, the other two lines—tangent to the envelope—intersect at right angles. Thus the circle is the orthoptic locus of the envelope. It is known that the orthoptic locus is a circle only for central conics and for curves of class three, order four, having l_∞ as double tangent at the circular points.[2] The above envelope must, therefore, be of order four, l_∞ being associated with diameters through both I and J, and touching the envelope at J and I. For all other diameters, the two associated lines cannot possibly coincide. The curve, then, has no other double tangents. Using the Plücker equations:

$$n = m(m - 1) - 2t - 3i$$
$$m = n(n - 1) - 2d - 3k$$

[1] Johnson, *Modern Geometry*, p. 211.

[2] Hilton, *Plane Algebraic Curves*, pp. 169–174.

and

$$i = 3n(n-2) - 6d - 8k$$

for $n=4$, $m=3$, and $t=0$, it is found that $i=0$, $d=0$, and $k=3$. Therefore, the envelope is a three-cusped hypocycloid.

It will now be shown which hypocycloid this is. By the method indicated on page 331, the lines associated with A_1R_1 are found to be A_1S_1 and R_1S_1; with A_2R_2, A_2S_2 and R_2S_2; with A_3R_3, A_3S_3 and R_3S_3.

It can readily be proven that the lines R_3S_3, A_1S_1, R_2S_2 are concurrent on K_1; R_2S_2, A_3S_3, R_1S_1 on K_3; and R_1S_1, A_2S_2, R_3S_3, on K_2. Therefore, the required hypocycloid must be tangent to the three sides of the triangle $K_1K_2K_3$, to its three altitudes, and to l_∞ twice at I and J. Then this hypocycloid must coincide with the Simson line envelope of points on the circumcircle of $K_1K_2K_3$ with respect to that triangle.

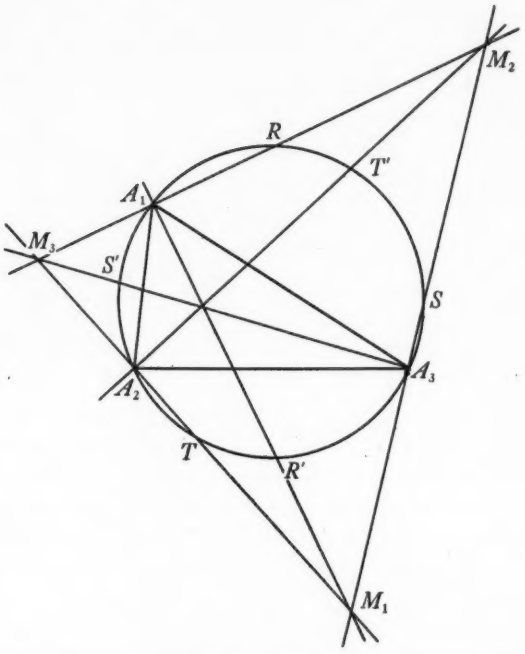
Theorem 17. As a point varies on the hypocycloid envelope of the Simson lines of the base triangle, the ortholines envelope the circumcircle of the base triangle and another three-cusped hypocycloid counted twice.

As the orthopole varies on a curve of order four, the ortholines have as envelope a curve of class eight. By Theorem 14, the circumcircle is part of this envelope. It will now be shown that the residual portion is a three-cusped hypocycloid, counted twice.

As a line moves parallel to itself, the two lines associated with it always pass through a fixed point on the circle. Also, the two lines associated with a tangent are always coincident. Now in every set of parallel lines, there will be two tangents to the circle at points T_1 and T_2, say. Then PT_1 and PT_2, each counted twice, will be associated with the tangents at the respective points, T_1 and T_2. Furthermore, PT_1 and PT_2 are perpendicular.

Since one of the three lines that has a point on the Simson line envelope as orthopole is always tangent to the circle, the other two will always be a certain line counted twice. Also, since l_∞ is associated with the tangents to the circle at both I and J, l_∞ is a double tangent. In a given direction, then, there can be drawn only one tangent. And, from what was said above, any two tangents that are perpendicular must intersect on the circumcircle of the base triangle. The envelope consequently is a curve of class three, counted twice, having the circumcircle as orthoptic locus. The line at infinity being the only double tangent, it follows, as in Theorem 16, that the curve is a hypocycloid of three cusps.

The hypocycloid is determined as follows. The lines associated with a line parallel to any side of the base triangle pass through the opposite vertex. Let a and a' be tangents parallel to A_2A_3; b and b', to A_1A_2; and c and c', to A_1A_3. Let the respective points of contact be R, R'; S, S'; T, T'. Then each of the following lines is tangent to the required hypocycloid: A_1R, A_1R', A_2T, A_2T', A_3S, A_3S'. It may readily be proven that A_1R, A_3S', and A_2T are concurrent on M_3; A_1R, A_2T', and A_3S, on M_2; and A_1R', A_2T, and A_3S, on M_1. The required



hypocycloid must coincide with the Simson line envelope of points on the circumcircle of triangle $M_1M_2M_3$ with respect to that triangle, each of the six lines mentioned above being such Simson lines.

VECTORIAL TREATMENT OF CERTAIN ALGEBRAIC THEOREMS

By T. C. ESTY, Amherst College

I. Consider the homogeneous linear equations

$$a_1x + b_1y + c_1z = 0, \tag{1}$$

$$a_2x + b_2y + c_2z = 0. \tag{2}$$

Let us define three vectors, α, β, ρ, as follows:

$$\alpha = a_1i + b_1j + c_1k, \quad \beta = a_2i + b_2j + c_2k, \quad \rho = xi + yj + zk, \tag{3}$$

where i, j, k form a right-handed system of mutually perpendicular unit vectors.

Equations (1), (2) may then be written as

$$\alpha \cdot \rho = 0, \tag{1'}$$

$$\beta \cdot \rho = 0. \tag{2'}$$

Equations (1′), (2′) are satisfied simultaneously by any vector ρ which is perpendicular to both α and β; that is, when ρ is given by

$$\rho = t\alpha \times \beta, \tag{4}$$

where t is any scalar.

Substituting for ρ, α, β their values from (3), expanding the vector product, and equating coefficients of i, of j, and of k, we obtain

$$x = t(b_1c_2 - b_2c_1), \quad y = t(c_1a_1 - c_2a_1), \quad z = t(a_1b_2 - a_2b_1), \tag{5}$$

showing that

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{6}$$

By putting $z = -1$ in (1) and (2) we obtain the system

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \end{aligned} \tag{7}$$

whose solution, as given by (6) when $z = -1$, is

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{8}$$

II. Consider the homogeneous linear equations,

$$a_1x + b_1y + c_1z = 0, \tag{1}$$

$$a_2x + b_2y + c_2z = 0, \tag{2}$$

$$a_3x + b_3y + c_3z = 0, \tag{3}$$

of which at least two, say (2) and (3), are independent. If we put

$$\begin{aligned} \alpha &= a_1i + b_1j + c_1k, & \beta &= a_2i + b_2j + c_2k, \\ \gamma &= a_3i + b_3j + c_3k, & \rho &= xi + yj + zk, \end{aligned} \tag{4}$$

then the given equations may be written in the form

$$\alpha \cdot \rho = 0, \tag{1′}$$

$$\beta \cdot \rho = 0, \tag{2′}$$

$$\gamma \cdot \rho = 0. \tag{3′}$$

We propose to find a necessary and sufficient condition that the last three equations have a common solution other than $\rho = 0$; that is, that the given equations have a common solution other than $x = y = z = 0$.

Let us first assume that there *is* such a value of ρ which satisfies (1′), (2′) and (3′). It follows that this ρ is perpendicular to each of the vectors α, β, γ, and hence that α, β, and γ are coplanar. This requires that

$$\alpha \cdot \beta \times \gamma = 0, \tag{5}$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{6}$$

This is a *necessary* condition.

Next, let us assume that (6), and hence (5), are fulfilled. Solving (2′) and (3′) for ρ in the form

$$\rho = t\beta \times \gamma \quad (\beta \text{ not parallel to } \gamma) \tag{7}$$

as in section I, we find by substitution in (1′),

$$\alpha \cdot \rho = t\alpha \cdot \beta \times \gamma,$$

and hence, by virtue of (5), $\alpha \cdot \rho$ vanishes identically. Thus $\rho = t\beta \times \gamma$, in which t is any scalar whatever, is a common solution of (1′), (2′) and (3′).

This proves that (5) is a *sufficient* condition that the equations have a common solution other than $\rho = 0$.

It follows that (6) is a necessary and sufficient condition that equations (1), (2) and (3) have a common solution other than $x = y = z = 0$.

One form of the solution may be found directly from (7), namely

$$x = t(b_2c_3 - b_3c_2), \quad y = t(c_2a_3 - c_3a_2), \quad z = t(a_2b_3 - a_3b_2), \tag{8}$$

where t is any scalar whatever.

By putting $z = -1$ in (1), (2), (3) we obtain the system

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \\ a_3x + b_3y &= c_3, \end{aligned} \tag{9}$$

and we see that (6) is a necessary condition that equations (9) have a common solution. If $a_2b_3 - a_3b_2 \neq 0$ the solution is unique; and it can be found from (8) by putting $z = -1$ and substituting the resulting value of t in the expressions for x and y.

III. Consider the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{1}$$

We shall assume, as we may without loss of generality, that the signs have been so adjusted as to make the quantities d_1, d_2, d_3 all positive.

Let us now define three vectors α, β, γ as follows:

$$\alpha = a_1 i + b_1 j + c_1 k, \quad \beta = a_2 i + b_2 j + c_2 k, \quad \gamma = a_3 i + b_3 j + c_3 k. \tag{2}$$

The equations (1) may be written in the equivalent form

$$\alpha \cdot \rho = d_1, \quad \beta \cdot \rho = d_2, \quad \gamma \cdot \rho = d_3. \tag{3}$$

If we regard α, β, γ, ρ as position vectors with a common origin at 0, these equations represent three planes which are perpendicular to α, β, γ, respectively, and at distances from 0 given by d_1/a, d_2/b, d_3/c, where a, b, c are the lengths of α, β, γ.

We now introduce the familiar formula

$$\rho(\alpha \cdot \beta \times \gamma) = \beta \times \gamma(\alpha \cdot \rho) + \gamma \times \alpha(\beta \cdot \rho) + \alpha \times \beta(\gamma \cdot \rho) \tag{4}$$

in which α, β, γ are supposed to be any three non-coplanar vectors. On the supposition, then, that the vectors α, β, γ in (2) are non-coplanar, that is, that $\alpha \cdot \beta \times \gamma \neq 0$, we have from (3) and (4)

$$\rho = \frac{d_1 \beta \times \gamma + d_2 \gamma \times \alpha + d_3 \alpha \times \beta}{\alpha \cdot \beta \times \gamma}. \tag{5}$$

This gives the position vector of the point common to planes (3). Replacing ρ by $xi+yj+zk$ and α, β, γ by their values in (2) we get

$$xi + yj + zk = \frac{d_1 \begin{vmatrix} i & j & k \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + d_2 \begin{vmatrix} i & j & k \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} + d_3 \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{6}$$

Expanding the determinants in the numerator it is found that the i-terms are

$$[d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)]i = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} i$$

with similar results for the j- and k-terms.

Equating coefficients of like vectors in (6) we obtain

$$(7)\qquad x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Just as (5) furnishes a single value of ρ when $\alpha\cdot\beta\times\gamma\neq 0$, so will equations (1) have a single solution when

$$(8)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

As a numerical example of the foregoing we shall solve the equations

$$\begin{aligned} x+\ 6y-\ 5z &= 23,\\ -\ 3x+8y-4z &= 1,\\ 7x-10y+10z &= 0. \end{aligned}$$

We put

$$\alpha = i+6j-5k,\qquad \beta = -3i+8j-4k,\qquad \gamma = 7i-10j+10k$$

and find

$$\beta\times\gamma = 40i+2j-26k,\ \gamma\times\alpha = -10i+45j+52k,\ \alpha\cdot\beta\times\gamma = 182.$$

We also have $d_1=23$, $d_2=1$, $d_3=0$.

Substituting these values in (5) we get

$$\rho = xi+yj+zk = \frac{23(40i+2j-26k)+(-10i+45j+52k)}{182} = 5i+\tfrac{1}{2}j-3k,$$

Whence $x=5$, $y=\frac{1}{2}$, $z=-3$.

Special cases which can arise under the supposition that $\alpha\cdot\beta\times\gamma$ *is* equal to zero, that is when

$$(9)\qquad \rho(0) = d_1\beta\times\gamma + d_2\gamma\times\alpha + d_3\alpha\times\beta,$$

may be considered under the following heads:

(1) When α, β, and γ are coplanar, without further restrictions.

Here there are two possibilities; (*a*) when the right side of (9) is *not* equal to zero, and (*b*) when it *is* equal to zero.

Under (*a*), there is no solution. In fact equations (3) show that the three planes are perpendicular to the plane of α, β, γ, and these planes intersect by pairs in three parallel lines whose direction is that of $\alpha\times\beta$.

Under (b),

$$d_1\beta \times \gamma + d_2\gamma \times \alpha + d_3\alpha \times \beta = 0. \tag{10}$$

To interpret this result we note that γ is in the plane of α and β; hence we may put

$$\gamma = l\alpha - m\beta. \tag{11}$$

Then $d_1\beta\times\gamma = ld_1\beta\times\alpha = -ld_1\alpha\times\beta$, $d_2\gamma\times\alpha = -md_2\beta\times\alpha = md_2\alpha\times\beta$ and (10) becomes $(-ld_1+md_2+d_3)\alpha\times\beta=0$, or since $\alpha\times\beta\neq 0$,

$$d_3 = ld_1 - md_2. \tag{12}$$

Substituting from (11) and (12) in the equation $\gamma\cdot\rho = d_3$, the equation of that plane becomes

$$l(\alpha\cdot\rho - d_1) - m(\beta\cdot\rho - d_2) = 0, \tag{13}$$

which is the equation of a plane through the intersection of the two planes $\alpha\cdot\rho = d_1$ and $\beta\cdot\rho = d_2$. Hence planes (3) *meet in one line*, which is parallel to $\alpha\times\beta$, and every point of this line is a solution of equations (3).

The corresponding values of x, y, z are therefore solutions of equations (1), and we say that these equations have an infinite number of solutions. To find the relations between the constants in (1) which exist in this case, we substitute the values of α, β, γ in (10) and expand the vector products. Then collecting the i-, j-, and k-terms separately, we get

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} i + \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} j + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} k = 0, \tag{14}$$

which requires that the three determinants vanish separately. In addition to these conditions we have, of course, since $\alpha\cdot\beta\times\gamma = 0$:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{15}$$

It is interesting to note that when equations (1) represent three planes with a common line of intersection, the equations are not independent, for by (11)

$$a_3 = la_1 - ma_2, \quad b_3 = lb_1 - mb_2, \quad c_3 = lc_1 - mc_2, \tag{11'}$$

and by (12)

$$d_3 = ld_1 - md_2. \tag{12'}$$

These relations show that the third of the given equations can be obtained by multiplying the first by a number l and subtracting from the result the second

multiplied by a number m. The process of finding the values of l and m will be illustrated in the following example.

Solve

$$\begin{aligned} 2x - 3y + 4z &= 5, \\ 3x + y - 2z &= 1, \\ -5x - 9y + 14z &= 7. \end{aligned}$$

Here $\alpha = 2i - 3j + 4k$, $\beta = 3i + j - 2k$, $\gamma = -5i - 9j + 14k$, and

$$\alpha \cdot \rho = 5, \quad \beta \cdot \rho = 1, \quad \gamma \cdot \rho = 7. \tag{3'}$$

Also $\alpha \cdot \beta \times \gamma = 0$, $\beta \times \gamma = -4i - 32j - 22k$, $\gamma \times \alpha = 6i + 48j + 33k$, $\alpha \times \beta = 2i + 16j + 11k$, so that

$$d_1\beta \times \gamma + d_2\gamma \times \alpha + d_3\alpha \times \beta = 5\beta \times \gamma + \gamma \times \alpha + 7\alpha \times \beta = 0.$$

Since no two of the vectors α, β, γ are parallel and since $d_1\beta \times \gamma + d_2\gamma \times \alpha + d_3\alpha \times \beta = 0$, the planes (3′) must meet in one line, and the given equations have an infinite number of solutions.

From the first of equations (11′) we have $2l - 3m = -5$, and from (12′) $5l - m = 7$. Solving these two equations, we find $l = 2$, $m = 3$. In fact, if we multiply the first of the given equations by 2 and subtract three times the second, we obtain the third.

If the equations of the line of intersection are desired, we may proceed as follows.

Let us first find the point in which the line pierces the plane of i and j. This point of the line may be regarded as the intersection of the three planes $\alpha \cdot \rho = 5$, $\beta \cdot \rho = 1$ and $k \cdot \rho = 0$. Denoting the vector of their common point by σ, and using the formula

$$\rho = \frac{d_1\beta \times \gamma + d_2\gamma \times \alpha + d_3\alpha \times \beta}{\alpha \cdot \beta \times \gamma}.$$

we have

$$\sigma = \frac{5\beta \times k + k \times \alpha}{\alpha \cdot \beta \times k} = \frac{8i - 13j}{11}.$$

Since the line is parallel to $\alpha \times \beta$, its equation will have the form $\rho = \sigma + t\alpha \times \beta$, where t is a variable scalar. Thus

$$xi + yj + zk = \frac{8}{11}i - \frac{13}{11}j + t(2i + 16j + 11k)$$

whence

$$\frac{x - \frac{8}{11}}{2} = \frac{y + \frac{13}{11}}{16} = \frac{z}{11}.$$

(2) When α, β, γ are coplanar, as before, but in addition, two of them, β and γ, (say) are parallel.

Here $\beta \times \gamma = 0$, and from (9)

$$\rho(0) = d_2\gamma \times \alpha + d_3\alpha \times \beta. \tag{16}$$

We have two possible cases; (*a*) when the right side is *not* equal to zero, and (*b*) when the right side *is* equal to zero.

Under (*a*) there is no solution. Two of the planes are parallel and are cut by the third plane in two lines which are parallel to $\alpha \times \beta$. Furthermore, since $\beta \times \gamma = 0$, we have

$$\frac{a_2}{a_3} = \frac{b_2}{b_3} = \frac{c_2}{c_3}. \tag{17}$$

Under (*b*),

$$d_2\gamma \times \alpha + d_3\alpha \times \beta = 0 \tag{18}$$

To interpret this, we first note that β and γ are supposed to be parallel. Now if they agree in sense, the vectors $\gamma \times \alpha$ and $\alpha \times \beta$ differ in sense. On the other hand, if β and γ differ in sense, then $\gamma \times \alpha$ and $\alpha \times \beta$ agree in sense. Under the latter supposition (18) is impossible, since d_2 and d_3 are positive. But under the former supposition, (18) becomes

$$d_2c - d_3b = 0, \text{ or } d_2/b = d_3/c. \tag{19}$$

Now d_2/b and d_3/c are the distances from 0 of the planes $\beta \cdot \rho = d_2$ and $\gamma \cdot \rho = d_3$, respectively, and since these distances are equal, the two planes coincide. All points of the line in which these coincident planes are cut by the third plane are solutions of (3), and hence equations (1) have in this case an infinite number of solutions.

It is easy to determine the conditions under which equations (1) present this case, for, since β is parallel to γ, and of the same sense, we may write

$$\beta = n\gamma, \text{ whence } n = b/c = d_2/d_3. \tag{20}$$

Also

$$a_2i + b_2j + c_2k = n(a_3i + b_3j + c_3k), \tag{21}$$

whence by (20) and (21)

$$a_2/a_3 = b_2/b_3 = c_2/c_3 = d_2/d_3. \tag{22}$$

(3) When all three vectors α, β, γ are parallel. Then, separately,

$$\beta \times \gamma = 0, \ \gamma \times \alpha = 0, \ \alpha \times \beta = 0. \tag{23}$$

In this case all of the planes (3) are parallel and in general there is no solution. We see from (23) that this situation arises when

$$a_2/a_3 = b_2/b_3 = c_2/c_3, \ a_3/a_1 = b_3/b_1 = c_3/c_1, \text{ and } a_1/a_2 = b_1/b_2 = c_1/c_2. \tag{24}$$

If, however, α, β, and γ agree in sense, and, in addition, $d_1/a = d_2/b = d_3/c$, then all three planes coincide, and, as in 2(b) the groups (24) of equal ratios include as a fourth member d_2/d_3, d_3/d_1, and d_1/d_2 respectively.

IV. To express the square of a determinant of the third order as a determinant of the same order.

Consider

$$(1) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2.$$

Introducing the vectors

$$\alpha = a_1 i + a_2 j + a_3 k,\ \beta = b_1 i + b_2 j + b_3 k,\ \gamma = c_1 i + c_2 j + c_3 k.$$

we may write (1) as

$$\begin{aligned} (\alpha\cdot\beta \times \gamma)(\alpha\cdot\beta \times \gamma) &= \alpha\cdot\beta \times [\gamma(\alpha\cdot\beta \times \gamma)], \\ &= \alpha\cdot\beta \times [\gamma(\alpha\cdot\beta \times \gamma) - \alpha(\gamma\cdot\beta \times \gamma)], \end{aligned}$$

since $\gamma\cdot\beta\times\gamma = 0$. Then

$$(2) \qquad (\alpha\cdot\beta \times \gamma)(\alpha\cdot\beta \times \gamma) = \alpha\cdot\beta \times [(\beta \times \gamma) \times \gamma \times \alpha],$$

by the formula for a vector triple product; or

$$\begin{aligned} (\alpha\cdot\beta \times \gamma)(\alpha\cdot\beta \times \gamma) &= \alpha \times \beta\cdot[(\beta \times \gamma) \times (\gamma \times \alpha)] \\ &= (\alpha \times \beta)\cdot(\beta \times \gamma) \times (\gamma \times \alpha) \end{aligned}$$

which is equivalent to

$$(3) \qquad \begin{vmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \end{vmatrix}.$$

V. To express the product of two determinants of the third order as a single determinant of the same order.

Consider the product

$$(1) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Introduce the vectors

$$(2) \qquad \begin{aligned} &\alpha = a_1 i + a_2 j + a_3 k, \quad \beta = b_1 i + b_2 j + b_3 k, \quad \gamma = c_1 i + c_2 j + c_3 k, \\ &\lambda = l_1 i + l_2 j + l_3 k, \quad \mu = m_1 i + m_2 j + m_3 k, \quad \nu = n_1 i + n_2 j + n_3 k. \end{aligned}$$

Then the product (1) may be expressed as

$$
\begin{aligned}
(3)\quad & (\alpha\cdot\beta\times\gamma)(\lambda\cdot\mu\times\nu)\\
&= \alpha\cdot\beta\times[\gamma(\lambda\cdot\mu\times\nu)]\\
&= \alpha\cdot\beta\times[\mu\times\nu(\gamma\cdot\lambda)+\nu\times\lambda(\gamma\cdot\mu)+\lambda\times\mu(\gamma\cdot\nu)]\\
&= \alpha\cdot\beta\times\mu\times\nu(\gamma\cdot\lambda)+\alpha\cdot\beta\times\nu\times\lambda(\gamma\cdot\mu)+\alpha\cdot\beta\times\lambda\times\mu(\gamma\cdot\nu)\\
&= \alpha\cdot[\mu(\beta\cdot\nu)-\nu(\beta\cdot\mu)](\gamma\cdot\lambda)+\alpha\cdot[\nu(\beta\cdot\lambda)-\lambda(\beta\cdot\nu)](\gamma\cdot\mu)\\
&\quad +\alpha\cdot[\lambda(\beta\cdot\mu)-\mu(\beta\cdot\lambda)](\gamma\cdot\nu)\\
&= (\alpha\cdot\mu)(\beta\cdot\nu)(\gamma\cdot\lambda)-(\alpha\cdot\nu)(\beta\cdot\mu)(\gamma\cdot\lambda)+(\alpha\cdot\nu)(\beta\cdot\lambda)(\gamma\cdot\mu)-(\alpha\cdot\lambda)(\beta\cdot\nu)(\gamma\cdot\mu)\\
&\quad +(\alpha\cdot\lambda)(\beta\cdot\mu)(\gamma\cdot\nu)-(\alpha\cdot\mu)(\beta\cdot\lambda)(\gamma\cdot\nu).
\end{aligned}
$$

Or

$$
(4)\qquad (\alpha\cdot\beta\times\gamma)(\lambda\cdot\mu\times\nu)=\begin{vmatrix}\alpha\cdot\lambda & \alpha\cdot\mu & \alpha\cdot\nu\\ \beta\cdot\lambda & \beta\cdot\mu & \beta\cdot\nu\\ \gamma\cdot\lambda & \gamma\cdot\mu & \gamma\cdot\nu\end{vmatrix}
$$

$$
=\begin{vmatrix}a_1l_1+a_2l_2+a_3l_3 & a_1m_1+a_2m_2+a_3m_3 & a_1n_1+a_2n_2+a_3n_3\\ b_1l_1+b_2l_2+b_3l_3 & b_1m_1+b_2m_2+b_3m_3 & b_1n_1+b_2n_2+b_3n_3\\ c_1l_1+c_2l_2+c_3l_3 & c_1m_1+c_2m_2+c_3m_3 & c_1n_1+c_2n_2+c_3n_3\end{vmatrix}.
$$

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE SOLUTIONS OF THE EQUATION $\binom{x+1}{y}=\binom{x}{y+1}$.

By W. RANDOLPH CHURCH, Hamden, Conn.

In the June–July issue of this Monthly, F. Underwood[1] solved the problem of determining all integral values of x and y which satisfy the equation

$$
(1)\qquad \binom{x}{y+1}=\binom{x+1}{y}.
$$

It is the purpose of the present note to bring out certain aspects of the subject which may be of interest. In particular, a different solution will be outlined in which it will not be necessary to distinguish different cases, and which will be

[1] This Monthly, vol. 38 (1931), pp. 351–354. The problem was proposed by Norman Anning, ibid., vol. 37 (1930), p. 508.

better adapted to the present needs than the one referred to. The relationship between the solutions of (1) and the well known numbers of Fibonacci will be brought out. And finally, it will be shown that the solutions form a recurring series, and the recurrence relations will be given.

Solution of Equation (1).

The equation (1) is equivalent to

$$x^2 - 3xy + y^2 - 2y - 1 = 0. \tag{2}$$

Making the substitution

$$\begin{aligned} x &= (3\xi + 4\eta - 6)/5 \\ y &= (2\xi + \eta - 4)/5 \end{aligned} \tag{3}$$

we obtain the equation

$$\xi^2 + \xi\eta - \eta^2 = -1. \tag{4}$$

The expression on the left hand side of (4) is the norm of an integer in the quadratic field $K(\sqrt{5})$:

$$N[\xi + \eta\omega] = \xi^2 + \xi\eta - \eta^2,$$

where $\omega = (1+\sqrt{5})/2$. The fundamental unit in this field is ω, and its norm is -1. From these facts it follows that the solutions of (4) are coextensive with the odd powers of ω. Thus, the complete solution of (4) is given by $\xi = \xi_m$, $\eta = \eta_m$, where

$$\omega^m = \xi_m + \eta_m\omega, \quad m = 2n + 1, \tag{5}$$

and n is any integer.

Expanding $\omega^m = (1+\sqrt{5})^m/2^m$ in (5) and equating rational and irrational parts, we find

$$(2\xi_m + \eta_m)2^{2n} = \sum_{i=0}^{n} \binom{2n+1}{2i} 5^i \tag{6}$$

and

$$\eta_m 2^{2n} = \sum_{i=0}^{n} \binom{2n+1}{2i+1} 5^i. \tag{7}$$

Setting $5^i = (2^2+1)^i$, expanding, reducing and solving these relations we obtain[1]

$$\xi_m = \sum_{i=0}^{n-1} \binom{n+i}{2i+1}, \tag{8}$$

[1] The reduction leading to (9) can be accomplished conveniently by using the fourth formula of (36) on p. 253 of Netto, *Combinatorik*, 1927. Expression (8) may then be obtained by using $\xi_{i+1} = \eta_i$, which is proved below.

$$\eta_m = \sum_{i=0}^{n} \binom{n+i}{2i}. \tag{9}$$

These expressions (8) and (9) give all integral solutions of (4), where $m=2n+1$; we will show that all integral solutions of (2) correspond to odd values of n.

For, from (6)

$$(2\xi_m + \eta_m)4^n \equiv 1 \pmod 5$$

or

$$2\xi_m + \eta_m \equiv (-1)^n \pmod 5$$

so that

$$2\xi_m + \eta_m - 4 \equiv (-1)^n + 1 \pmod 5.$$

If we denote the values of x and y corresponding to $\xi=\xi_m$ and $\eta=\eta_m$ as x_m and y_m respectively, comparing the last congruence above with (3) we have the condition

$$0 \equiv (-1)^n + 1 \pmod 5;$$

whence y_m is integral if and only if $n=2k+1$. Adding the two equations (3), we get $x+y=\xi+\eta-2$, so that x_m is integral if and only if y_m is integral.

The complete solution of (1) is then given by (8) and (9) where $m=2n+1$, $n=2k+1$. It can be shown without difficulty that the solutions of (1) correspond to the positive values of m. Therefore, unless otherwise specified, m will henceforth be a positive integer, congruent to 3, modulo 4.

Relations to the Numbers of Fibonacci.[1]

It can be shown that

$$\begin{aligned} \xi_i &= u_{i-1} \\ \eta_i &= u_i \end{aligned} \qquad i = 1, 2, \cdots, \tag{10}$$

where u_i is the ith Fibonacci Number. For, from (5)

$$\omega^{i+1} = \xi_{i+1} + \eta_{i+1}\omega,$$

and multiplying (5) by ω and reducing by $\omega^2=\omega+1$,

$$\omega^{i+1} = \eta_i + (\xi_i + \eta_i)\omega.$$

These give, upon eliminating ω^{i+1} and equating powers of ω:

$$\xi_{i+1} = \eta_i$$

$$\eta_{i+1} = \xi_i + \eta_i$$

[1] The facts regarding the numbers of Fibonacci made use of in this paper may be found in Lucas, *Théorie des Nombres*, vol. 1, Paris, 1891, pp. 3 f., 127.

Using these relations together, we have

$$\xi_{i+1} = \xi_i + \xi_{i-1} \tag{11}$$

$$\eta_{i+1} = \eta_i + \eta_{i-1}. \tag{12}$$

But the numbers of Fibonacci may be defined by

$$u_0 = 0,\ u_1 = 1,\ u_{i+1} = u_i + u_{i-1}$$

and using $m = 0, 1, 2$ in (5) we obtain

$$\xi_0 = 1 \qquad \eta_0 = 0$$
$$\xi_1 = 0 \qquad \eta_1 = 1$$
$$\xi_2 = 1 \qquad \eta_2 = 1$$

from which (10) follows.

All solutions of (1) can then be written

$$\begin{aligned} x_m &= (3u_{m-1} + 4u_m - 6)/5, \\ y_m &= (2u_{m-1} + u_m - 4)/5. \end{aligned} \tag{13}$$

By applying the relations

$$u_i = u_{i-1} + u_{i-2} \tag{14}$$

and

$$u_{k+j} = u_{j+1}u_k + u_j u_{k-1} \tag{15}$$

these can be brought into the form

$$\begin{aligned} x_m &= (u_{m+3} + u_{m+1} - 1)/5 - 1, \\ y_m &= (u_{m+1} + u_{m-1} + 1)/5 - 1. \end{aligned} \tag{16}$$

From (14) the relation

$$u_{i+3} + u_{i+1} = 5u_i + u_{i-1} + u_{i-3} \tag{17}$$

easily follows. Using this with (16) the following can be shown by induction to hold:

$$\begin{aligned} x_m &= u_m + u_{m-4} + \cdots + u_3 - 1 \\ y_m &= u_{m-2} + u_{m-6} + \cdots + u_1 - 1. \end{aligned} \tag{18}$$

If we denote by $U_{k,n}$ the sum of the first $n+1$ Fibonacci numbers whose subscripts are congruent to k, modulo 4, that is

$$U_{k,n} = u_k + u_{4+k} + \cdots + u_{4n+k},\ k = 0, 1, 2, 3,$$

then

$$U_{k,n} = u_{2(n+1)}u_{2n+k},\ k = 0, 1, 2, 3. \tag{19}$$

For, by using (15) we have

$$u_{4n+k+4} + u_{2(n+1)}u_{2n+k} = u_{2n+3}u_{2n+2+k} + u_{2n+2}u_{2n+1+k} + u_{2n+1}u_{2n+k}$$

and the right hand side of this reduces to

$$u_{2(n+2)}u_{2(n+1)+k}$$

upon combining and applying (14). Hence (19) may be easily established by induction; in fact, if

$$U_{k,n} = u_{2(n+1)}u_{2n+k}$$

then

$$U_{k,n} + u_{4n+k+4} = u_{2(n+1)}u_{2n+k} + u_{4n+k+4}$$

or, by the preceding and (14),

$$U_{k,n+1} = u_{2(n+2)}u_{2(n+1)+k}.$$

Referring now to (18) it is clear that

$$\begin{aligned} x_m &= U_{3,(m-3)/4} - 1 \\ &= u_{(m+1)/2}u_{(m+3)/2} - 1, \end{aligned} \tag{20}$$

and

$$\begin{aligned} y_m &= U_{1,(m-3)/4} - 1 \\ &= u_{(m+1)/2}u_{(m-1)/2} - 1. \end{aligned} \tag{21}$$

Also, using (18) and (14),

$$\begin{aligned} x_m - y_m &= u_m - u_{m-2} + u_{m-4} - u_{m-6} + \cdots + u_3 - u_1 \\ &= u_{m-1} + u_{m-5} + \cdots + u_2 \\ &= U_{2,(m-3)/4} \\ &= u^2_{(m+1)/2}. \end{aligned} \tag{22}$$

Again, using (18) and (14)

$$\begin{aligned} x_m + y_m &= u_m + u_{m-2} + u_{m-4} + u_{m-6} + \cdots + u_3 + u_1 - 2 \\ &= 3u_{m-2} + u_{m-3} + 3u_{m-6} + u_{m-7} + \cdots + 3u_1 + u_0 - 2 \\ &= 3y_m + U_{0,(m-3)/4} + 1, \end{aligned}$$

so that

$$\begin{aligned} x_m - 2y_m &= U_{0,(m-3)/4} + 1 \\ &= u_{(m+1)/2}u_{(m-3)/2} + 1 \\ &= u^2_{(m-1)/2}, \end{aligned} \tag{23}$$

since $u_i^2 = u_{i-1}u_{i+1} + (-1)^i$.

Using (3) we have

$$\begin{aligned} x_m + y_m &= \xi_m + \eta_m - 2 \\ &= u_{m-1} + u_m - 2 \\ &= u_{m+1} - 2. \end{aligned} \tag{24}$$

Recurrence Relations.

The recurrence relations are

$$\begin{aligned} x_m &= 8(x_{m-4} - x_{m-8}) + x_{m-12} \\ y_m &= 8(y_{m-4} - y_{m-8}) + y_{m-12}. \end{aligned} \tag{25}$$

By using (14) successively it can easily be shown that

$$8u_{i-4} = u_i + u_{i-4} + u_{i-8}.$$

The relations (25) follow immediately upon substituting from (18) in the right hand side and using the relation just mentioned; for example

$$\begin{aligned} 8(x_{m-4} - x_{m-8}) + x_{m-12} &= 8u_{m-4} + (u_{m-12} + \cdots + u_3 - 1) \\ &= u_m + u_{m-4} + u_{m-8} + \cdots + u_3 - 1. \\ &= x_m. \end{aligned}$$

The following relation connects the y_i with the x_i of the same and one lower rank:

$$3y_m = x_m + x_{m-4}. \tag{26}$$

Writing (26) as

$$3y_m - x_m = x_{m-4}$$

and substituting in the left hand side of this from (18), the result reduces to x_{m-4} upon applying (14) successively.

The following table contains the first few values of x and y which satisfy (1).

m	3	7	11	15	19	23	27
x	1	14	103	713	4894	33551	229969
y	0	5	39	272	1869	12815	87840

A Geometric Paradox

By C. H. Rowe, Trinity College, Dublin

Consider the quadric surface that is represented by the general equation of the second degree in four homogeneous coordinates. It is well known that we must impose three independent conditions on the coefficients in order that the

quadric should reduce to a pair of planes. However, the following argument *seems* to show that two conditions are sufficient.

By imposing the condition that the discriminant of the equation should vanish we ensure that the quadric is degenerate, and that we can transform to a new system of homogeneous coordinates so that the transformed equation contains only three of the four variables. One further condition is sufficient to ensure that this transformed equation represents a pair of planes. The total number of conditions that we need is therefore two.

A Note by the Editor

Professor Rowe writes: "Encouraged by the interest shown in Professor Coolidge's *Paradox* in the American Mathematical Monthly, I venture to send you another paradox which contains a different kind of fallacious argument about the number of conditions that are required for a certain geometrical property."

Now will some one furnish the explanation for Professor Rowe's paradox?

R.E.G.

RATIONAL RIGHT TRIANGLES

By H. T. R. AUDE, Colgate University

This paper presents a method of finding integral solutions of the equation $x^2+y^2=z^2$, or of obtaining rational right triangles. It uses one of the acute angles of the triangle as a function and a few trigonometric formulas. These help to show the necessary and sufficient conditions and assist in establishing a systematic procedure.

The problem may be limited to finding only the primitive right triangles where x, y and z, the integers for the sides, are relatively prime. Let A be the angle opposite to y, then it will be possible to find integers for x, y, and z, only if $\sin A$ and $\cos A$ are both rational. From the formula $\tan A/2=\sin A/(1+\cos A)$ it is seen that $\tan A/2$ will also be rational; and since A is an acute angle of the triangle, $A/2<\pi/4$. Conversely, if $\tan A/2$ is a positive proper fraction, then A is acute and belongs to a right triangle, whose sides are the integers x, y, z, where y/z and x/z are respectively equal to $2\sin A/2\cos A/2$ and $\cos^2 A/2-\sin^2 A/2$. If $\tan A/2$ is taken equal to n/m, where n and m are integers relatively prime and $n<m$, then a set of values for x, y, z are

$$m^2-n^2,\quad 2mn,\quad m^2+n^2. \tag{1}$$

In order to obtain all rational primitive right triangles with a hypotenuse h less than a certain integer p we select systematically values for $\tan A/2=n/m$ such that the integers m and n, relatively prime, satisfy the conditions $m^2+n^2<p^2$. All the proper fractions thus formed may be arranged in two groups:

(I) those for which m and n are not both odd numbers, and (II) those for which m and n are both odd. If A and B denote the two acute angles of the right triangle and if $\tan A/2 = n/m$, then $\tan B/2 = (m-n/(m+n)$. And, furthermore, it becomes evident upon examination that if n/m is a fraction of group (I) n and m not both odd, then $(m-n)/(m+n)$ is a fraction of group (II) with numerator and denominator both odd; and the converse also holds. In fact this relation between the fractions of the two groups is involutory. The fractions in group (I) when selected for $\tan A/2$ lead by formulas (1) to the same right triangles as the fractions in group (II) which represent in some order corresponding values of $\tan B/2$. Therefore, by omitting entirely group (II) and choosing for $\tan A/2 = n/m$ only those proper fractions where m and n are not both odd, there is obtained once and once only all the primitive right triangles with $h < p$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Nomographie. By H. Fréchet and H. Roullet. Paris, Librarie Armand Colin, 1928. Paper, 208 pages. 10.50 fr.

Handbook of Statistical Nomographs, Tables, and Formulas By J. W. Dunlap and A. K. Kurtz. Yonkers, N. Y., World Book Company, 1932. x+164 pages.

The Physical Significance of the Quantum Theory. By F. A. Lindemann. Oxford, the Clarendon Press, 1932. viii+148 pages. $2.50.

National Council of Teachers of Mathematics. The Fifth Yearbook: The Teaching of Geometry. xii+206 pages. The Sixth Yearbook: Mathematics in Modern Life. xii+126 pages. The Seventh Yearbook: The Teaching of Algebra. xii+180 pages. New York, Teachers College, 1930, 1931, 1932.

Mathematical Nuts for Lovers of Mathematics. By Samuel I. Jones. Nashville, Published by the Author, 1932. xii+340 pages.

Partial Differential Equations of Physics. By H. Bateman. New York, Macmillan, 1932. xxii+522 pages.

Modern Junior Mathematics. By W. W. Hart. Three books, with teacher's manuals. New York, D. C. Heath, 1931.

C-D Machine Correlation Chart. Designed by E. E. Cureton and J. W. Dunlap. 25 sheets, $16\frac{1}{2}'' \times 11''$. New York, Macmillan, 1932. $1.00.

Die Liesche Theorie der partiellen Differentialgleichungen Erster Ordnung. By F. Engel and K. Faber. Leipzig, B. G. Teubner, 1932. xii+368 pages. 28 marks.

Freshman Mathematics; a course in the essentials of Algebra, Trigonometry, and Analytic Geometry. By H. L. Slobin and W. E. Wilbur. New York, Long and Smith, 1932. xviii+438 pages.

Algebra. By Oskar Perron. I. Die Grundlagen. Göschens Lehrbucherei, I Gruppe, Band 8. Berlin, Walter de Gruyter, 1932. viii+302 pages. 12.80 marks.

Dynamics of Engine and Shaft. By Ralph E. Root. New York, John Wiley, 1932. viii+184 pages. $3.00.

Le Calcul des Différences finies et ses applications. By Alfred Henry. Paris Hermann, 1932. 212 pages. 50 francs.

Outline of the History of Mathematics. By R. C. Archibald. Bulletin 18 of the Society for Promotion of Engineering Education, 1932. 54 pages.

Catalog of an Exhibition at Columbia University to commemorate the one hundredth anniversary of the birth of Lewis Carroll. New York, Columbia University Press, 1932. 154 pages. 35 cents.

Philosophische Versuch über die Wahrscheinlichkeit. By P. S. Laplace. Ostwalds Klassiker der Exakten Wissenschaften, bo. 233. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1932. viii+212 pages, 9.60 marks.

College Algebra. By H. P. Pettit and P. Luteyn. New York, John Wiley and Sons, 1932. viii+284 pages. $1.90.

The Calculus. By H. H. Dalaker and H. E. Hartig. Second Edition. New York, McGraw Hill Book Company, 1932. viii+276 pages. $2.25. (From the preface to the Second Edition: This edition is substantially the same as the first. A considerable number of exercises and a chapter on differential equations have been added.)

REVIEWS

The Foundations of Mathematics. By F. P. Ramsey. New York, Harcourt, Brace, & Co., 1931. xviii+292 pages.

This is a collection of essays in mathematics, mathematical logic, and philosophy, including the most important of the author's previously published works, and a number of papers, some of them fragmentary, which were still unpublished at the time of the author's death in 1930.

The first of these essays, which was originally published in the *Proceedings of The London Mathematical Society* in 1926, sets forth the author's proposal for revision of the system of logic contained in Whitehead and Russell's *Principia Mathematica.* Three important changes are advocated, namely, abandonment of the principle that x and y are identical (or equal) when every propositional function satisfied by x is also satisfied by y, abandonment of the principle that every class is determined by a propositional function, and a modification of the theory of types designed to avoid the much discussed axiom of reducibility.

Of these proposals, the first seems to be open to serious objection. For if x and y are any two things which have all their properties in common, and if we allow that x has the property of being identical with x, then we must allow that y also has the property of being identical with x, that is, that $y=x$. The possibility that x may not be a definable object, and therefore that the property of being identical with x may not be a definable property, is clearly irrelevant.

And the second proposal is open to an entirely similar objection, for if α is a class, then the propositional function $\hat{x}\epsilon\alpha$ determines α, and also the propositional function, "$\hat{x}$ has all the elementary properties which are common to all the members of α," determines α. Of course, there may be such things as undefinable classes, and if α is an undefinable class then there is no definable propositional function which determines α. But this, as we understand it, is not denied by the authors of *Principia Mathematica*.

The third proposal, however, is worthy of more serious consideration. Mr. Ramsey observes that the contradictions with which the theory of types is intended to deal fall into two classes, firstly what we may call the mathematical paradoxes (for example the paradox of Burali-Forti) which can be expressed entirely by means of the symbols of formal logic, and secondly what we may call the epistemological paradoxes (for example Richard's paradox) which require the use of the verb "means." Two hierarchies of types, of rather different sorts, are required, one of them for the sake of avoiding the mathematical paradoxes, and the other to avoid the epistemological paradoxes. And it is the second of these two hierarchies which renders the axiom of reducibility necessary. Mr. Ramsey's proposal is to replace this second hierarchy of types by another hierarchy of types, which he defines, and under which an axiom of reducibility is not necessary. And he then gives, in terms of this new hierarchy, a solution of the epistemological paradoxes which depends on an ambiguity in the meaning of "means."

Distrust of the axiom of reducibility is, of course, widespread, being shared even by the authors of *Principia Mathematica*, and there seems to be no doubt of the desirability of a theory which avoids this axiom. But we cannot agree with Mr. Ramsey, that the reason for the desirability of avoiding it is that the axiom is not a tautology in the sense of Wittgenstein, or that it is desirable or necessary that all the axioms of logic should be tautologies. For the notion of a tautology in this sense depends for its intuitive significance on the identification of $(x).\ \phi x$ with the logical product of all possible values of ϕx, and this identification seems to be doubtful, because the nature of a logical product is such that its meaning cannot be understood without first understanding the meaning of each proposition which enters into the logical product.

And certainly the notion of a tautology loses much of its connotation of "necessary" when we discover that the axiom of infinity is a tautology if it be true, but a contradiction if it be false.

It is worth remarking that the proof that the axiom of infinity is a tautology (if true) depends, not, as the author seems to imply, on his proposed revision of

the notion of identity, but wholly on the introduction of propositional functions in extension. This notion of a propositional function in extension is certainly legitimate, but it seems doubtful whether the distinction can successfully be maintained between ordinary propositional functions and propositional functions in extension.

In the second essay of the book, the author defends his treatment of mathematical logic against the formalism of Hilbert and the intuitionism of Brouwer and Weyl. The third essay, which is devoted to the solution of a particular case of the Entscheidungsproblem, contains also a theorem of the theory of classes which is not without interest on its own account. And the remaining papers are, for the most part, on subjects which are philosophical, rather than mathematical, in character. All of them are well worth reading, for the sake of the author's insight into the questions of which he treats, and for their power of stimulating thought on the part of the reader.

ALONZO CHURCH

Vectorial Mechanics. By Louis Brand. New York, John Wiley and Sons, Inc., 1930. xvii+544 pages.

This is a very carefully planned textbook in which the fundamental problems of statics, kinematics and dynamics of a particle and of rigid bodies are treated methodically by the vector method. An introductory chapter of thirty-eight pages gives a clear account of the vector algebra necessary for problems of statics. Four chapters (132 pages) are devoted to statics and following this a chapter of fourteen pages is devoted to vector analysis. Chapters 7 to 10, inclusive (154 pages) are devoted to flexible cables and kinematics and these are followed by four chapters (174 pages) on dynamics. A final chapter treats problems of impact. Every chapter has examples fully and carefully worked out and closes with a summary of the important results. The typography is excellent and there seem to be very few misprints. It is clear that the book will be very useful to students of mechanics.

The author states in his preface that the book has been written with a view to engineering applications, but the mathematics involved has not been shirked. An engineer mastering this book will know a lot of geometry and algebra. It is perhaps inevitable that certain compromises have to be made. Thus in the very beginning a vector quantity is defined as one involving the idea of direction, but this is a vague idea. It is necessary for the author to say later that the zero vector is not a proper vector because it does not have a determinate direction The zero vector is in some respects the most fundamental vector of all since every vector equation expresses that a combination of vectors is the zero vector and it is very unfortunate to have, or to convey, the impression that there is something improper about this fundamental vector. Again in vector analysis it might perhaps be emphasized somewhat more that a vector can not be differentiated without considering a reference frame. It is very important for a student

to grasp the obvious fact that such things as velocity, momentum, energy are not properties of a system, but expressions of the relation of the system to a reference frame. However, your reviewer has sufficiently expressed his opinions on these matters in the book "Theoretical Mechanics" (written in collaboration with President Ames) and he need say no more than that he can strongly recommend the present work to beginning students and teachers alike. The wise teacher will tell his students about other good textbooks, such as Routh's five volumes or Hamel's fine book "Elementare Mechanik."

F. D. MURNAGHAN.

Fourier'sche Reihen. By J. Wolff. Groningen, P. Noordhoff N. V., 1931. 60 pages.

This booklet has for its purpose the exposition of certain well known properties of Fourier Series. The text is divided into three sections each supplemented by exercises which total over a hundred. The first two sections deal with Riemann integrable functions and are intended for students of pure as well as of applied mathematics, while the last section deals with summable functions and is intended for students interested primarily in pure mathematics.

The first section contains proofs of the usual type of pointwise and of uniform convergence under simple hypotheses on the function; nothing is said about the degree of the convergence. In the second section, a discussion of the properties of the first arithmetic mean of Féjer is followed by more general theorems which have for their basis the least square property of the partial sum of the Fourier Series. Theorems previously proved are extended to the case of summable functions in the first part of the third section. These are followed by general propositions on sequences of measurable sets and functions which form a basis for theorems concerning the Fourier coefficients and the validity of the Fourier expansion of suitably restricted summable functions. Misprints which occur are few in number and easily recognizable as such.

J. M. EARL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3557. *Proposed by Robert E. Moritz, University of Washington*

Given a free flow of water from a vertical circular aperture, flowing partially full. The radius of the aperture is r, the depth of the water above the center of the aperture is a, the coefficient of entry is k. Required the rate of discharge under the action of gravity.

3558. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Of a plane triangle given one side, the length of the bisector of the opposite angle, and the altitude on the given side; state and prove the ruler-compass construction of the triangle.

3559. *Proposed by George A. Yanosik, New York University.*

Variable circles are drawn having any point on a central conic and one of its foci as ends of a diameter. Prove that the envelope of these circles is the auxiliary circle of the central conic.

3560. *Proposed by Frank Morley, Johns Hopkins University.*

In a Euclidean space, perpendiculars from the vertices of a regular tetrahedron to a plane meet the plane at the points represented by the complex numbers x_i, $i = 1, 2, 3, 4$. Show that the four points obey the relation

$$\sum^{6}(x_i - x_j)^2 = 0.$$

3561. *Proposed by Emmanuel Wad, Baltimore, Md.*

The number 12345678 is not divisible by 11, but by placing the eight figures in different orders we can form other numbers which are divisible by 11. Determine how many such numbers can be formed.

SOLUTIONS

3233. [1927, 45]. *Proposed by A. A. Bennett, Lehigh University.*

Describe geometrically the continuous function represented by the Fourier series, $\Sigma(\cos mt)/m^2$, where m is restricted to positive integers prime to 6. (This question arose in the theory of three-phase alternating currents.)

Solution by Robert E. Moritz, University of Washington

The series is obviously absolutely convergent and will remain so if it is augmented by the terms corresponding to values of m which are not prime to 6.

Denote the sum of the given series by y, then we have

$$y = y_1 - y_2 - y_3 + y_6, \tag{1}$$

where, m taking in succession the values 1, 2, 3, etc.,

$$y_1 = \sum\frac{\cos mt}{m^2}, \quad y_2 = \sum\frac{\cos 2mt}{(2m)^2}, \quad y_3 = \sum\frac{\cos 3mt}{(3m)^2}, \quad y_6 = \sum\frac{\cos 6mt}{(6m)^2}, \tag{2}$$

and we may obviously write,

$$(3)\qquad \begin{cases} y_2 = \dfrac{1}{4}\sum \dfrac{\cos mu}{m^2}, & u = 2t; \\ y_3 = \dfrac{1}{9}\sum \dfrac{\cos mv}{m^2}, & v = 3t; \\ y_6 = \dfrac{1}{36}\sum \dfrac{\cos mw}{m^2}, & w = 6t. \end{cases}$$

Now the sum y_1 is well-known.[1] Its value is

$$(4)\qquad y_1 = \frac{1}{12}[3(t - \overline{2k+1}\pi)^2 - \pi^2], \qquad 2k\pi \leqq t \leqq 2(k+1)\pi;$$

hence also

$$(5)\qquad \begin{cases} y_2 = \dfrac{1}{4\cdot 12}[3(2t - \overline{2k+1}\pi)^2 - \pi^2], & k\pi \leqq t \leqq (k+1)\pi; \\ y_3 = \dfrac{1}{9\cdot 12}[3(3t - \overline{2k+1}\pi)^2 - \pi^2], & 2k\pi/3 \leqq t \leqq 2(k+1)\pi/3; \\ y_6 = \dfrac{1}{36\cdot 12}[3(6t - \overline{2k+1}\pi)^2 - \pi^2], & k\pi/3 \leqq t \leqq (k+1)\pi/3. \end{cases}$$

Let us consider the positive t-axis divided into intervals of $\pi/3$ each, beginning at the origin. The values of k in y_1, y_2, y_3, y_6, depend on the interval in which t lies and may be readily computed. Let n be one less than the number of the interval in which t lies. The value of k in y_1 will then be the quotient obtained on dividing n by 6, the values of k in y_2 and y_3 will be the integral parts of the quotients obtained on dividing n by 3 and 2 respectively, and the value of k in y_6 will be n itself. If we denote by n_i the integral part of the quotient of n divided by i, we have from (1), (4), and (5)

$$y = \frac{1}{12}[3(t - \overline{2n_6+1}\pi)^2 - \pi^2] - \frac{1}{4\cdot 12}[3(2t - \overline{2n_3+1}\pi)^2 - \pi^2]$$

$$- \frac{1}{9\cdot 12}[3(3t - \overline{2n_2+1}\pi)^2 - \pi^2] + \frac{1}{36\cdot 12}[3(6t - \overline{2n+1}\pi)^2 - \pi^2]$$

which on reducing gives

$$(6)\qquad y = A\pi t + A'\pi^2,$$

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[1] Weber, H., *Die Partiellen Differentialgleichungen*, 6th Edition, Vol. 1, p. 76.

where

$$A = (-6n_6 + 3n_3 + 2n_2 - n - 1)/6,$$

$$A' = [36n_6(n_6 + 1) - 9n_3(n_3 + 1) - 4n_2(n_2 + 1) + n(n + 1) + 4]/36.$$

The quantities A and A' may be further simplified. By definition

$$n_6 = (n - r_6)/6, \quad n_3 = (n - r_3)/3, \quad n_2 = (n - r_2)/2,$$

where r_6, r_3, r_2, are the remainders resulting from the division of n by 6, 3, and 2 respectively. These values substituted in A and A' give

$$y = A\pi t + (Bn + C)\pi^2, \tag{7}$$

where

$$A = (r_6 - r_3 - r_2 - 1)/6, \quad B = -A/3,$$

$$C = [r_6^2 - r_3^2 - r_2^2 + 4 - (6r_6 - 3r_3 - 2r_2)]/36,$$

and n is one less than the interval in which t lies.

Since each remainder r_i is limited to 0 or an integer less than i, we see that A, B, and C, admit each of at most 6 different values which are repeated cyclically for each successive set of six intervals.

Let us tabulate the successive values of r_6, r_3, r_2, in the first six intervals, and compute the corresponding values of A, B, and C.

Interval	1	2	3	4	5	6
$n =$	0	1	2	3	4	5
$r_2 =$	0	1	0	1	0	1
$r_3 =$	0	1	2	0	1	2
$r_6 =$	0	1	2	3	4	5
$6A =$	-1	-2	-1	$+1$	$+2$	$+1$
$18B =$	$+1$	$+2$	$+1$	-1	-2	-1
$18C =$	$+2$	$+1$	-1	-2	-1	$+1$
$18(Bn + C) =$	$+2$	$+3$	$+1$	-5	-9	-4

We can now write down the value of the sum of the given series for any given value of t.

If

$$0 \leqq t \leqq \pi/3, \qquad y = -\pi t/6 + \pi^2/9;$$

$$\pi/3 \leqq t \leqq 2\pi/3, \qquad y = -\pi t/3 + \pi^2/6;$$

$$2\pi/3 \leqq t \leqq \pi, \qquad y = -\pi t/6 + \pi^2/18;$$

$$\pi \leqq t \leqq 4\pi/3, \qquad y = +\pi t/6 - 5\pi^2/18;$$

$$4\pi/3 \leqq t \leqq 5\pi/3, \qquad y = +\pi t/3 - \pi^2/2;$$

$$5\pi/3 \leqq t \leqq 2\pi, \qquad y = +\pi t/6 - 2\pi^2/9; \text{ etc.}$$

Geometrically considered the series represents an infinite number of line-segments whose slopes are limited to four values of A and whose intercepts vary indefinitely with t.

3298. [1927, 537]. *Proposed by W. A. Thompson, Supt. of Schools, Webster, South Dakota.*

Given two circles of unequal sizes, such circles intersecting each other, to draw a secant cutting both circles, such that the chords of the circles are equal to each other and equal to the segment of the line between the two chords.

Solution by Otto Dunkel, Washington University.

Let the two circles have the points O_1, O_2 as centers; let their radii be R, r; let $O_1O_2 = a$; and let $2x$ be the length of the required chord. We find at once by inspection of a figure

$$[(R^2 - x^2)^{1/2} \pm (r^2 - x^2)^{1/2}]^2 + 16x^2 = a^2,$$

where the plus sign refers to the case where the required secant cuts within the segment O_1O_2, and the minus sign to the case where the secant cuts the extension of O_1O_2. This equation is easily solved for x^2, and it will be shown how to determine the number and character of the real constructions given by this solution for x.

The given problem will be considered as including equal circles as well as non-intersecting circles. If the given circles are tangent, the common tangent gives a trivial construction; if a required secant passes through the center of the smaller circle, we shall call this a diameter secant; if it cuts within the segment O_1O_2 we shall call it a W secant; if outside, an E secant. Two constructions for which the secants are symmetrically placed with respect to O_1O_2 will be considered as a single construction.

If we set $x^2 = y$, $a^2 = z$, then

$$z = 14y + R^2 + r^2 \pm 2[(R^2 - y)(r^2 - y)]^{1/2}, \quad R \geqq r, \tag{1}$$

where the plus sign gives a W construction, the minus sign, an E construction. With a horizontal y-axis and a vertical z-axis, the above equation is that of an hyperbola, for which the part above the diameter $z = 14y + R^2 + r^2$ gives the W solutions, while the part below gives the E solutions. If $R = r$, the curve degenerates into the two straight lines

$$z = 12y + 4R^2, \quad z = 16y, \tag{2}$$

which meet in the point $(R^2, 16R^2)$. Since r^2 and R^2 enter symmetrically in (1) and in the given diameter, we obtain the equations of the asymptotes and the center

$$z = 12y + 2(R^2 + r^2), \quad z = 16y, \quad [(R^2 + r^2)/2, \ 8(R^2 + r^2)] \tag{3}$$

by replacing $2R^2$ in the above results by R^2+r^2. The z-intercepts of (1) are $(R-r)^2$ and $(R+r)^2$, while those of the asymptotes are 0 and $2(R^2+r^2)$. Since $0 \leqq (R-r)^2 < (R+r)^2 \leqq 2(R^2+r^2)$, the position of the curve between the asymptotes is located. One branch of the curve is beyond $y=(R^2+r^2)/2 \geqq r^2$, and this part can give no real construction. Moreover, the part on the negative side of $y=0$ gives no real construction. Thus the real constructions are confined to that part of one branch which lies within the triangle formed by the two straight lines in (3) and the z-axis.

There is a vertical tangent at $[r^2, R^2+15r^2]$, the point of separation of the W and E parts of the curve. By setting $dz/dy=0$, we find a horizontal tangent at (y_1, z_1), where

$$y_1 = [(12 - 7\sqrt{3})R^2 + (12 + 7\sqrt{3})r^2]/24,$$
$$z_1 = 4[(2 - \sqrt{3})R^2 + (2 + \sqrt{3})r^2],$$

and this fixes the greater limit for a^2 in certain cases. In the following separation of the cases, the single letter E denotes a single real construction of this type, while E, W denotes two real constructions, one of each type, etc.

Case I. $R \leqq 7r$.

$$0 \leqq (R - r)^2 \leqq a^2 < (R + r)^2,\ E;\ (R + r)^2 \leqq a^2 < R^2 + 15r^2,\ E,\ W;$$
$$R^2 + 15r^2 \leqq a^2 \leqq z_1,\ W,\ W.$$

If $R=r$, the last interval reduces to a point.

Case II. $7r<R<(7+4\sqrt{3})r$.

$$0 < (R - r)^2 \leqq a^2 < R^2 + 15r^2,\ E;\ R^2 + 15r^2 \leqq a^2 < (R + r)^2,\ W;$$
$$(R + r)^2 \leqq a^2 \leqq z_1,\ W,\ W.$$

Case III. $R \geqq (7+4\sqrt{3})r$.

$$0 < (R - r)^2 \leqq a^2 < R^2 + 15r^2,\ E;\ R^2 + 15r^2 \leqq a^2 \leqq (R + r)^2,\ W.$$

The trivial constructions are easily recognized in the above scheme; when $a^2=R^2+15r^2$, there is a diameter construction; when $a^2=z_1$, the two W constructions coincide in the same secant.

We may make use of a different unknown. Let M be the middle point of O_1O_2; N, the intersection of the radical axis of the two circles with this line, $a=4u$, $MN=m$. If the secant cuts the radical axis in the point P, then MP is perpendicular to the secant and P is the mid-point of the segment between the circles (see the solutions of problem 3288 [1927, 491] in this Monthly, vol. 36 (1929), pp. 108–110). Let the absolute value of the power of N with respect to either circle be p^2; the power is positive if the circles do not intersect and negative if they do. Then the secant can be constructed from $NP=t$, which satisfies the equation

$$t^4 + (m^2 - 3u^2 \pm p^2)t^2 \pm p^2m^2 = 0,$$

where

$$2am = R^2 - r^2, \ \pm 4a^2p^2 = (a^2 - R^2 - r^2)^2 - 4r^2R^2, \ \ 3x^2 = t^2 \pm p^2.$$

The equation in x may be written in the form

$$3x^4 + (m^2 - 3u^2 \mp p^2)\, x^2 \pm u^2p^2 = 0,$$

or

$$48x^4 + [8(R^2 + r^2) - 7a^2]x^2 \pm a^2p^2 = 0,$$

3482. [1931, 170]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If the circumscribed and the inscribed spheres of a tetrahedron are concentric, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Solution by Wallace Smith, New River State College, Montgomery, W. Va.

Sphere S is circumscribed about and sphere s is inscribed in the tetrahedron $ABCD$, each having the center O. Since planes ABD, BDC, ADC, and ABC are tangent to sphere s, they are at equal distances from the center of sphere S. Planes equidistant from the center of a sphere intersect the sphere in equal circles. Therefore, triangles ABD, ABC, ADC, and BCD are inscribed in equal circles. The chords AD, AB, and BD are each common to two equal circles. It follows at once that $\angle ACD = \angle ABD$, $\angle BCA = \angle BDA$, and $\angle BCD = \angle BAD$. Therefore, $\angle BCD + \angle BCA + \angle ACD$ equals $\angle BAD + \angle BDA + \angle ABD$ equals two right angles.

A similar proof applies to the other three vertices.

Therefore, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Also solved by V. F. Ivanoff, A. Pelletier, J. Rosenbaum, Otto J. Ramler, and F. Underwood.

A Note by Otto Dunkel. In the above proof the fact that pairs of angles such as ACD and ABD are equal rather than supplementary follows from the fact that each of such face angles is acute. For, the center O of the inscribed sphere s is the intersection of the three planes bisecting the dihedral angles at AB, BC, CA. Thus, if the point of tangency of s with the face ABC is T_d, this point must lie within each angle of the triangle ABC, and, consequently, it must lie within the area of that triangle. Moreover, since O is also the center of S, T_d is the center of the circle circumscribing ABC and lying upon S. Since the center of the circumcircle lies within ABC, all its angles must be acute; and, similarly,

all the face angles of the tetrahedron must be acute. It was proved above that the four circumcircles are equal.

Let the other three faces be rotated into the plane of ABC so that the vertex D falls at D', D'', D''', where A and D', B and D'', C and D''', lie, respectively, on opposite sides of BC, CA, AB. As shown above the interior angles at D', D'', D''' are equal, respectively, to the angles A, B, C of triangle ABC. Also $\angle D''AC = C'$, $\angle D'''AB = B'$, where $A + B' + C' = 180°$; thus $D''AD'''$ is a straight line. Similarly, $D'''BD'$ and $D'CD''$ are straight lines. Hence $D'D''D'''$ is a triangle with C, A, B as the mid-points of its sides. Therefore BC is parallel to $D'''D''$, and $B' = B$, $C' = C$. Continuing in this way we see that the four triangles are congruent. Thus the tetrahedron $ABCD$ is such that its opposite edges are equal.

Such tetrahedrons may be constructed by taking any triangle $D'D''D'''$ which has only acute angles and reversing the above process.

3491. [1931, 339]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The internal bisectors of the angles subtended by the six edges of a tetrahedron at the centroid of the tetrahedron are such that the three lines joining the points on the pairs of opposite edges are concurrent.

Solution by F. Underwood, University College, Nottingham.

Let G be any point in space, and let E be the intersection with AB of the internal bisector of the angle at G of the triangle GAB. Let F, H, K, L, M be points on CD, AC, BD, AD, BC, respectively, determined with respect to G in a similar manner. Denote AG, BG, CG, DG by a, b, c, d, respectively. Then

$$\frac{AE}{a} = \frac{EB}{b}, \quad \frac{CF}{c} = \frac{FD}{d}. \tag{1}$$

Consider now the centroid of a system of particles of masses m/a, m/b, m/c, m/d, attached to A, B, C, D, respectively. Then by (1) E is the centroid of the particles at A and B, and F is the centroid of the particles at C and D. Hence the centroid of the four particles lies on EF. Similarly, it lies on HK and on LM; and therefore EF, HK, and LM are concurrent.

A Note by Otto Dunkel: This problem is closely related to the system of barycentric coordinates with $ABCD$ as the tetrahedron of reference, in which the coordinates α, β, γ, δ of a point P are defined as equal, or proportional, to the volumes of the tetrahedrons $PBCD$, $APCD$, $ABPD$, $ABCP$. The coordinates may be considered as positive if P is inside $ABCD$. If the plane of PCD cuts AB in P_{ab}, then

$$\frac{AP_{ab}}{P_{ab}B} = \frac{\text{vol. } APCD}{\text{vol. } PBCD} = \frac{\beta}{\alpha},$$

since the two tetrahedrons have the base PCD in common. The remaining edges are divided in a similar manner. The lines AP_{bc}, BP_{ca}, CP_{ab} meet in a point P_d, the intersection of DP with the face ABC. The planes PCD, PAB have obviously the points P, P_{ab}, P_{cd} in common and hence these three points lie in a straight line. There are similar results for the remaining two such pairs of points.

Conversely, it is easily seen that, if α, β, γ, δ are given and if the edges of $ABCD$ are divided as above, a point P is determined with these numbers as coordinates. It is also easily shown that if particles of masses α, β, γ, δ are attached to the vertices of $ABCD$, the center of mass of the particles at A and B is P_{ab}; the center of mass of those at A, B, C is P_d; the center of mass of the whole system is P.

As shown in the above solution, the point G determines a point P with the barycentric coordinates a^{-1}, b^{-1}, c^{-1}, d^{-1}. These coordinates are defined even in the case where G lies on an edge, but not at a vertex. In the latter case it is usual to take all the coordinates but one as zero: the geometric theorem is trivial.

If P_{ab}' is taken on AB so that it is the harmonic conjugate of P_{ab} with respect to A, B, we obtain, proceeding similarly, six new points, which are easily shown to lie in a plane. This is called the polar plane of P with respect to $ABCD$. In the problem P_{ab}' is determined by the external bisector of $\angle AGB$. The equation of the polar plane of P in this case is

$$a\alpha + b\beta + c\gamma + d\delta = 0,$$

and the coordinates of this plane may be taken as a, b, c, d.

Also solved by A. Pelletier, and J. Rosenbaum.

3493. [1931, 339]. *Proposed by Paul Wernicke, Washington, D.C.*

An opaque circular circumference with a small transparent gap G rotates about its center with uniform velocity. A luminous point L outside illuminates through G a wire soldered to the circumference at two points and forming a curve going through C. The point of this curve illuminated by the ray LG is always to lie on a perpendicular to CL at C. (a) Give the equation of the curve formed by the wire and the points where it is soldered to the circumference, and (b) Give the velocity of the illuminated point on CL.

Solution by Wm. B. Campbell, Rangoon, Burma.

Since the curve (P) to be derived passes through C, it will be found that the light source L lies in the plane of the circumference. Set $CL=b$, $CG=a$, $CP=r$, and $\angle GCP=\theta$; then it is easily seen from a figure that

$$(1) \qquad r = \frac{ab\cos\theta}{b - a\sin\theta}.$$

This is the equation of (P) with reference to CG. As θ increases from zero, r increases from a at G to its maximum value of $ab(b^2-a^2)^{-1/2}$ for which $\theta_1=\arcsin(a/b)$. It then decreases to a for which $\theta_2=2\arctan(a/b)$; it continues to de-

crease from this point and becomes zero when $\theta=\pi/2$. The part of the curve for $\pi/2<\theta\leqq\pi$ is symmetrical with respect to CG to the part already described. The part described so far resembles a cardioid; it is perpendicular to CG at C and the upper part has the slope b/a at G. The part of the curve inside the circle and illuminated by the ray of light through G is given by

$$\theta_2 \leqq \theta \leqq \pi - \theta_2, \quad \tan(\theta_2/2) = a/b; \tag{2}$$

and the remaining part of the curve for the interval $0\leqq\theta\leqq\pi$ lies outside the circle. For the interval $\pi<\theta\leqq 2\pi$ there is a loop inside the circle and also inside the part described above, tangent to it at C and beginning at G with a slope of $-b/a$. This loop is also symmetrical with respect to CG.

The velocity v of the spot of light on (P) is $(ds/d\theta)(d\theta/dt)$, or

$$v^2 = (\omega ab)^2[\cos^2\theta(b - a\sin\theta)^{-2} + (a - b\sin\theta)^2(b - a\sin\theta)^{-4}], \tag{3}$$

where ω is the angular velocity of rotation.

Also solved by Frank L. Wilmer and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Secretary W. D. Cairns of the Mathematical Association of America, with Mrs. Cairns, will spend July and early August in Scotland, England and France, the remainder of August in Belgium, Holland, the Rhine country and Switzerland, attending the International Congress of Mathematicians at Zurich in September. Their mail address will be in care of the American Express Company (10, Frederick St., Edinburgh, until July 6; 84 Queen St. E. C. 4, London until July 24; 11 Rue Scribe, Paris, until August 10) and then in care of the Congress at Zurich. The address for Association business will be Oberlin, Ohio, as usual.

On April 19, 1932, Professor Robert Courant of Göttingen delivered the first William Lowell Putnam Lecture of 1932 at Harvard University on the subject "Hyperbolic partial differential equations and wave mechanics."

On April 27, 1932, Professor R. L. Moore of Texas, delivered the second William Lowell Putnam Lecture of 1932 at Harvard University on the subject "The foundations of point set theory."

Professor Arnold Dresden of Swarthmore College gave a lecture at Oberlin College April 25, 1932, on "Logic, Mathematics and Reality" under the auspices of the departments of philosophy and mathematics.

Dr. Cornelius Lanczos, on leave from the University of Frankfort, who has been visiting professor of mathematics at Purdue University during 1931–32, has received a permanent appointment at the latter institution as professor of

mathematical physics. Professor Lanczos will be in residence only during the first semester of each year.

Dr. Hassler Whitney, National Research Fellow, will conduct a Seminar in Metric Spaces at Harvard University during the first term of 1932–33.

Professor Frederick Wood, of Hamline University, has been appointed head of the department of mathematics at the University of Nevada.

Dr. John J. Gergen has been reappointed a Benjamin Pierce Instructor at Harvard University for the year 1932–33.

Dr. Vladimir Seidel has been appointed a Benjamin Pierce Instructor at Harvard University for the year 1932–33.

The following men have been appointed Part Time Instructors at Harvard University for the year 1932–33. P. A. Adams, T. L. Downs, J. S. Frame, A. S. Galbraith, Z. I. Mosesson, L. T. Moston, F. H. Steen.

Dr. L. A. Bauer, Director Emeritus of the Department of Terrestrial Magnetism, Carnegie Institution of Washington, died on April 12, 1932. He was a charter member of the Association.

Professor C. G. Simpson of the department of mathematics, College of Engineering, Milwaukee, Wis., a charter member of the Association, died February 5, 1932.

The following seventy-six doctorates with mathematics or mathematical physics as major subject were conferred during 1931 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

D. B. Ames, Yale, June, *A resonance problem in celestial mechanics.*

F. S. Beale, Michigan, *On the solutions of systems of linear difference equations with polynomial coefficients.*

H. F. Bohnenblust, Princeton, April, *The absolute convergence of Dirichlet series.*

Samuel Borofsky, Columbia, April, *Expansion of analytic functions into infinite products.*

Sister Leonarda Burke, Catholic, June, minors in education and physics, *On a case of the triangles in- and circumscribed to a rational quartic curve with a line of symmetry.*

L. E. Bush, Ohio State, August, *Some properties of algebras without moduli.*

S. S. Cairns, Harvard, June, *The cellular division and approximation of regular spreads.*

Helen Calkins, Cornell, November, *Some implicit functional theorems.*

A. B. Cardwell, Wisconsin, June, major in physics, minor in mathematical physics, *Photoelectric and thermionic properties of metals.*

Evelyn T. Carroll, Cornell, October, *Systems of involutorial birational transformations contained multiply in special linear line complexes.*

Emily M. Chandler (Mrs. H. H. Pixley), Chicago, August, *Waring's theorem for fourth powers.*

Max Coral, Chicago, August, *The Euler-Lagrange multiplier rule for double integrals.*

A. T. Craig, Iowa, June, *On the distribution of certain statistics derived from small random samples.*

J. H. Dillon, Wisconsin, June, major in physics, minor in mathematics, *Photoelectric properties of zinc single crystals.*

J. E. Donohue, Columbia, March, *Concerning the geometry of the second derivative of a polygenic function.*

H. L. Dorwart, Yale, June, *Certain types of criteria for the irreducibility of polynomials.*

R. D. Douglass, Massachusetts Institute of Technology, June, minor in physics, *Stirling expansions derived by means of finite de la Vallee-Poussin summation.*

Ben Dushnik, Michigan, *On the Stieltjes integral.*

J. N. Eastham, Catholic, June, minors in civil engineering and physics, *The triangles in- and circumscribed to the tacnodal rational quartic curve with residual crunode.*

H. C. T. Eggers, Michigan, *The graduation of experimental data.*

J. M. Feld, Columbia, November, *Projective pedal transformations and birational contact transformations.*

N. C. Fisk, Michigan, *An investigation of surfaces in euclidean four-space by means of three-vectors.*

A. L. Foster, Princeton, January, *A continuous development of formal logic in finite terms.*

Sister Mary de Lellis Gough, Catholic, June, minors in education and physics, *On the condition for the existence of triangles in- and circumscribed to certain types of the rational quartic curve and having a common side.*

Eli Gourin, Columbia, May, *On irreducible polynomials in several variables which become reducible when the variables are replaced by powers of themselves.*

M. L. Hartung, Wisconsin, June, minor in mathematical physics, *On a family of integral equations with discontinuous kernels.*

T. W. Hatcher, Cornell, August, minor in engineering, *Symmetric strain in an infinite plate with a circular hole.*

R. A. Hefner, Chicago, August, *The condition of Mayer for discontinuous solutions of the Lagrange problem.*

C. T. Holmes, Harvard, June, *The approximation of harmonic functions in three dimensions by harmonic polynomials.*

H. N. Hubbs, Cornell, February, minor in physics, *Rational quintic surfaces without double curves.*

W. R. Hutcherson, Cornell, June, minor in physics, *Maps of certain cyclic involutions on two dimensional carriers.*

D. R. Inglis, Michigan, *Atomic problems in the perturbation theory of quantum mechanics.*

Sister Mary Cordia Karl, Johns Hopkins, June, *The projective theory of ortho-poles.*

Donald McDonough, Pennsylvania, June, *On the expansion of a certain type of determinant.*

W. O. Menge, Michigan, *A direct method of obtaining transformations to canonical forms.*

H. L. Miller, Cincinnati, June, *On the summability of non-integral orders of the double Fourier series.*

G. E. Moore, Illinois, June, minor in astronomy, *The four-termed theta identities arising from a generalization of the Weddle surface in* S_4.

C. B. Morrey, Harvard, June, *Invariant functions of conservative surface transformations.*

Sister Charles Mary Morrison, Catholic, June, minors in philosophy and physics, *The triangles in- and circumscribed to the biflecnodal rational quartic curve.*

M. E. Mullings, Cincinnati, June, minor in physics, *On Gibb's phenomenon in the double Fourier series.*

C. V. Newsom, Michigan, *On the behavior of entire functions in distant portions of the plane.*

E. G. Olds, Pittsburgh, June, *The nature of the distributions in small samples.*

A. L. O'Toole, Michigan, *On symmetric functions and symmetric functions of symmetric functions.*

L. J. Paradiso, Cornell, October, minor in economics, *Solutions of bounded variation of the Fredholm Stieltjes integral equation.*

W. V. Parker, Brown, June, *The addition formulas for hyperelliptic functions.*

Jesse Pierce, Michigan, *Logarithmic solutions of systems of differential equations and solutions in the vicinity of isolated singular points.*

H. H. Pixley, Chicago, August, *A problem in the calculus of variations suggested by a problem in economics.*

Saul Pollock, California (Berkeley), May, *Graphic representation in enumerative geometry.*

J. E. Powell, Chicago, August, *Edge conditions for multiple integrals in the calculus of variations.*

H. R. Pyle, California (Berkeley), May, *The conditions for conformality in the elliptic and hyperbolic geometries.*

Mina S. Rees, Chicago, March, *Division algebras associated with an equation whose group has four generators.*

B. J. Roberts, Iowa, August, minor in physics, *On continua in vectorial spaces.*

H. A. Robinson, Johns Hopkins, March, *Case of planar motion in which one centrode and one path are circles.*

S. L. Robinson, Iowa, August, minors in physics and philosophy, *A study of simple and multiple transitivity in topological spaces.*

A. E. Ross, Chicago, August, *On representation of integers by indefinite ternary quadratic forms.*

T. A. Rouse, Wisconsin, June, major in physics, minor in mathematics, *Characteristic X-ray absorption.*

R. G. Sanger, Chicago, June, *Functions of lines and the calculus of variations*

M. G. Scherberg, Minnesota, June, minor in physics, *The degree of convergence of a series of Bessel functions.*

Mabel F. Schmeiser, Ohio State, August, *Properties of arbitrary functions concerning approach to a straight line.*

C. Grace Shover, Ohio State, December, *On the class number and ideal multiplication in a rational linear associative algebra.*

James Singer, Princeton, April, *Three-dimensional manifolds and their Heegaard diagrams.*

P. K. Smith, Illinois, June, minor in physics, *On solutions of linear partial differential equations near singular points.*

T. L. Smith, Harvard, June, *The Birkhoff fluid theory of electricity.*

I. S. Sokolnikoff, Wisconsin, June, *On a solution of Laplace's equation with an application to the torsion problem for a polygon with reentrant angles.*

F. W. Sparks, Chicago, August, *Universal quadratic zero forms in four variables.*

R. C. Stephens, Iowa, August, minor in physics, *Continuous functions on abstract sets.*

W. T. Stratton, Washington (Seattle), August, minor in physics, *A study of general polar tangent curves.*

Mildred E. Taylor, Illinois, August, minor in physics, *A determination of the types of planar Cremona transformations with not more than 9 F-points.*

C. C. Torrance, Cornell, June, minor in physics, *On plane Cremona triadic characteristics.*

Sister Mary Felice Vaudreuil, Catholic, June, minors in education and physics, *Two correspondences determined by the tangents to a rational cuspidal quartic curve with a line of symmetry.*

C. C. Wagner, Michigan, *A statistical study of the dependence for four chance variables.*

W. G. Warnock, Illinois, May, minor in astronomy, *On the geometry of groups of line configurations.*

H. E. Wheeler, Chicago, June, *An application of symbolic methods to frequency arrays.*

S. S. Wilks, Iowa, June, minor in education, *On the distribution of statistics in samples from a normal population of two variables with matched sampling of one variable.*

R. P. Winch, Wisconsin, June, major in physics, minor in mathematics, *Photoelectric properties of silver.*

Yue-Kei Wong, Chicago, December, *Spaces associated with non-modular matrices with application to reciprocals.*

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twelfth regular meeting of the Southern California Section was held at State Teachers College, San Diego, on Saturday, March 26, 1932. Professor H. C. Van Buskirk presided.

The attendance was forty-five including the following twenty members of the Association: L. D. Ames, Harry Bateman, Clifford Bell, P. H. Daus, A. S. Gale, Raymond Garver, J. M. Gleason, J. D. Hill, G. H. Hunt, C. G. Jaeger, L. M. Klauber, G. R. Livingston, W. E. Mason, G. F. McEwen, G. E. F. Sherwood, H. M. Showman, S. E. Urner, H. C. Van Buskirk, Morgan Ward, H. C. Willett.

The meeting began with a luncheon at the Aztec restaurant on the new campus of the San Diego State Teachers College, after which it adjourned to Science Hall for a short business meeting and the program. The following officers were elected for the year 1932–33: Chairman, L. D. Ames, University of Southern California; Vice-chairman, O. W. Albert, University of Redlands; Program Committee, C. G. Jaeger, Pomona College, and Clyde Wolfe, California Institute of Technology. The next meeting was tentatively scheduled for March 18, 1933, at Pomona College.

The following program was presented:

1. "Mathematics and finance" by Julius Wangenheim, President of the Southern Title and Trust Company of San Diego, by invitation.

2. "A triangular arrangement of numbers leading to the graphical determination of Euler sequences" by L. M. Klauber, San Diego.

3. "The application of the law of mathematical probability to the behavior of gases in their pressure-volume-temperature relations" by Dr. A. Linhart, Riverside Junior College, by invitation.

4. "Personal reminiscences of Josiah Willard Gibbs" by Dean A. S. Gale, University of Rochester.

5. "Function-theoretic properties of a simple logarithmic transcendent" by Professor Morgan Ward, California Institute of Technology.

6. "A class of surfaces applicable to the sphere" by Professor C. G. Jaeger, Pomona College.

7. "On the approximate solution of certain equations" by Professor Raymond Garver, University of California at Los Angeles.

8. "On representation of integers by certain quadratic forms" by Dr. A. E. Ross, California Institute of Technology, by invitation.

Abstracts of certain of these papers follow, the numbers corresponding to the numbers of the titles.

2. The integers are arranged in a triangular order with 1 at the apex, the second line containing numbers 2 to 4, the third 5 to 9, and so forth. When the primes have been indicated, it is found that there are concentrations in certain vertical and diagonal lines, and amongst these the so-called Euler sequences